Derivation of Transfer Functions for a Fluid-Loaded, Multiple-Layer Thick Plate System
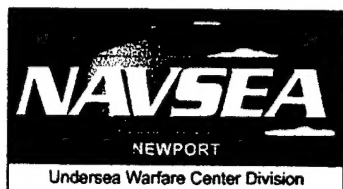
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PREFACE

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Transfer functions are mathematical expressions that relate two physical quantities of a system. When applied to mechanical systems, transfer functions model energy propagation through the medium of a structure. These transfer functions are useful in understanding system behavior, and they sometimes can be verified by measurements at different locations in the structure. This report derives six mathematical transfer functions of various thick plate systems. The equations of motion and stress are derived for one, two, and three thick plates with and without fluid loading on one surface. These models are validated with comparisons to thin plate equations of motion using very small plate thicknesses.

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DERIVATION OF TRANSFER FUNCTIONS FOR A FLUID-LOADED, MULTIPL-LAYER THICK PLATE SYSTEM

1. INTRODUCTION

Transfer functions are mathematical expressions that relate two physical quantities of a system. When applied to mechanical systems, transfer functions model energy propagation through the medium of a structure. This medium can be simple or complex, depending on the type and composition of the structure. Typically, partial differential equations describe some field quantity on the medium. A set of assumptions is made, and these partial differential equations reduce to ordinary differential equations and then to algebraic equations. The end result is an expression that is the quotient of two of the field quantities, frequently a system value at some location in the field divided by an energy input into the field. These transfer functions are useful in understanding system behavior, and they sometimes can be verified by measurements at different locations in the structure.

This report derives six mathematical transfer functions of various thick plate systems. First, two-dimensional equations of elasticity are derived and mathematical expressions for the plate displacements and spatial derivatives of the plate displacements are calculated. Second, the equations of motion of a single thick plate with a normal pressure load are then determined. Third, the single plate is coupled with a fluid whose dynamics are governed by the wave equation. This fluid contains an acoustic pressure that acts on the structure and changes the dynamic behavior of the system. The response of this system is calculated. Fourth, the equations of motion of two connected thick plates with a normal pressure load are determined. This double (thick) plate allows for two different types of plate materials to be attached together. Fifth, the fluid load is then coupled with the double thick plate and the response is calculated. Sixth, the equations of motion of three connected thick plates with a normal pressure load are determined. The triple (thick) plate allows for three different types of plate materials to be attached together. Seventh, the fluid load is then coupled to the triple thick plate and the response is calculated. In all cases, the thick plate equations of motion are verified by comparing their transfer functions to a second transfer function derived using the thin plate theory calculated with a very small plate thickness. This allows validation of the results using a second method of modeling the response.

2. EQUATIONS OF ELASTICITY

The transfer functions are derived by assuming that each thick plate is governed by the equation (reference 1)

$$\mu\nabla^2\mathbf{u} + (\lambda+\mu)\nabla\nabla\bullet\mathbf{u} = \rho\frac{\partial^2\mathbf{u}}{\partial t^2}, \tag{1}$$

where ρ is the density; λ and μ are the Lamé constants; t is time; $\bullet$ denotes a vector dot product; **u** is the Cartesian coordinate displacement vector expressed as

$$\mathbf{u} = \begin{Bmatrix} u_x(x,y,z,t) \\ u_y(x,y,z,t) \\ u_z(x,y,z,t) \end{Bmatrix},$$

(2)

with subscript x denoting the direction parallel to the plates, y denoting the direction into the plates, and z denoting the direction perpendicular to the plates. The modeled geometry and the coordinate system of the plates are shown in figure 1. The symbol ∇ is the gradient vector differential operator written in three-dimensional Cartesian coordinates (reference 2) as

$$\nabla = \frac{\partial}{\partial r} i_r + \frac{1}{r}\frac{\partial}{\partial \theta} i_\theta + \frac{\partial}{\partial z} i_z,$$

(3)

with i_x denoting the unit vector in the x-direction, i_y denoting the unit vector in the y-direction, and i_z denoting the unit vector in the z-direction; ∇^2 is the three-dimensional Laplace operator operating on vector **u** as

$$\nabla^2 \mathbf{u} = \nabla^2 u_x i_x + \nabla^2 u_y i_y + \nabla^2 u_z i_z,$$

(4)

with ∇^2 operating on scalar u as

$$\nabla^2 u_{x,y,z} = \nabla \bullet \nabla u_{x,y,z} = \frac{\partial^2 u_{x,y,z}}{\partial x^2} + \frac{\partial^2 u_{x,y,z}}{\partial y^2} + \frac{\partial^2 u_{x,y,z}}{\partial z^2},$$

(5)

and the term $\nabla \bullet \mathbf{u}$ is called the divergence and is equal to

$$\nabla \bullet \mathbf{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}.$$

(6)

The applied loading effects of the pressure in the absence of a fluid acting on the surface of the plate are modeled as a structural load in sections 3, 5, and 7. The applied loading effects of the pressure in the presence of a fluid acting on the surface of the plate are modeled as an acoustic load in sections 4, 6, and 8. Note that ρ, λ, and μ are the properties of each specific plate.

The displacement vector **u** is written as

$$\mathbf{u} = \nabla \phi + \nabla \times \vec{\psi},$$

(7)
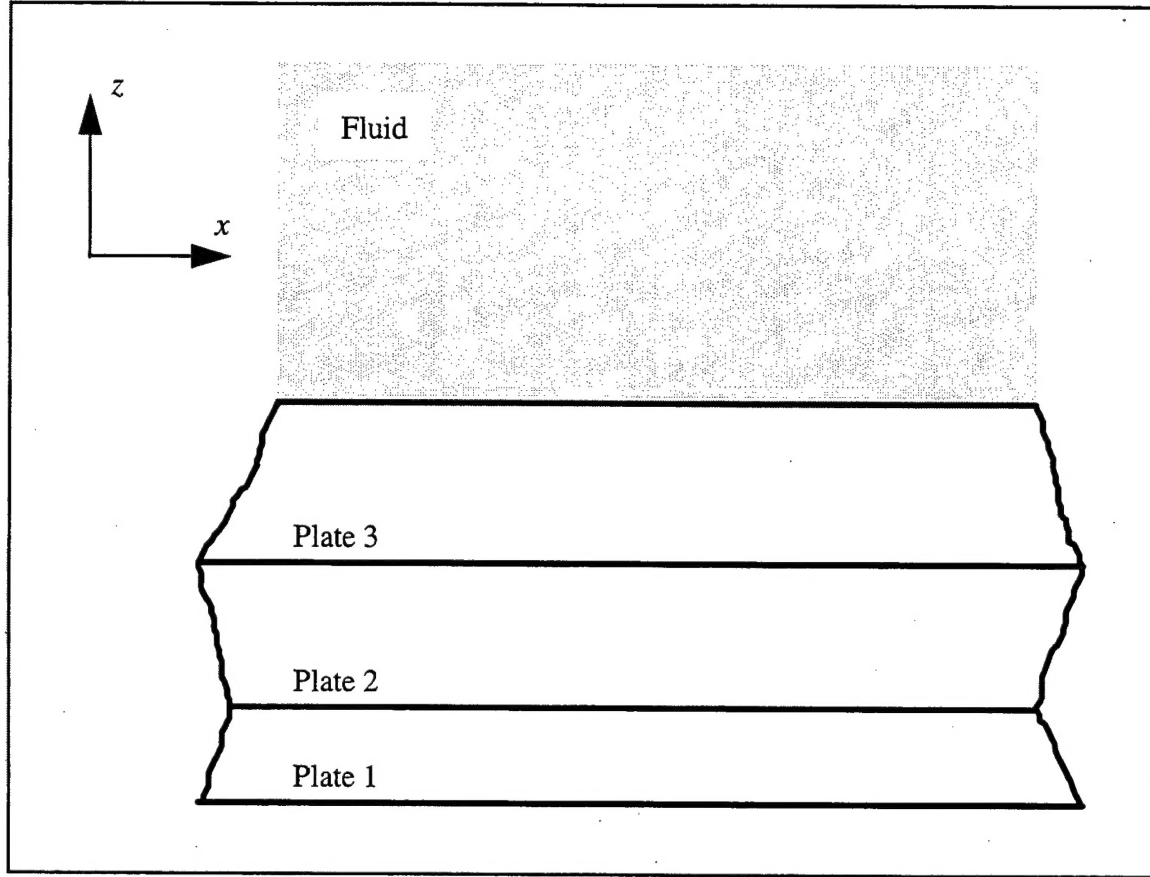
Figure 1. Modeled Geometry with Coordinate System

where ϕ is a dilatational scalar potential, $\times$ denotes a vector cross product, and $\bar{\psi}$ is an equivoluminal vector potential expressed as

$$\bar{\psi} = \begin{Bmatrix} \psi_x(x,y,z,t) \\ \psi_y(x,y,z,t) \\ \psi_z(x,y,z,t) \end{Bmatrix}.$$

(8)

The problem is now formulated as a two-dimensional response ($y \equiv 0$ and $\partial(\cdot)/\partial y \equiv 0$) problem. Expanding equation (7) and breaking the displacement vector into its individual nonzero terms yield

$$u_x(x,z,t) = \frac{\partial \phi(x,z,t)}{\partial x} - \frac{\partial \psi_y(x,z,t)}{\partial z},$$

(9)

and

$$u_z(x,z,t) = \frac{\partial \phi(x,z,t)}{\partial z} + \frac{\partial \psi_y(x,z,t)}{\partial x}.$$

(10)

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Equations (9) and (10) are next inserted into equation (1), which results in

$$c_d^2 \nabla^2 \phi(x,z,t) = \frac{\partial^2 \phi(x,z,t)}{\partial t^2},$$

(11)

and

$$c_s^2 \nabla^2 \psi_y(x,z,t) = \frac{\partial^2 \psi_y(x,z,t)}{\partial t^2},$$

(12)

on all material layers. Equation (11) is the dilatational component and equation (12) is the shear component of the displacement field (reference 3). Correspondingly, the constants c_d and c_s are the complex dilatational and shear wave speeds, respectively, and are determined by

$$c_d = \sqrt{\frac{\lambda + 2\mu}{\rho}},$$

(13)

and

$$c_s = \sqrt{\frac{\mu}{\rho}}.$$

(14)

The relationship of the Lamé constants to the compressional and shear moduli is shown as

$$\lambda = \frac{E\upsilon}{(1+\upsilon)(1-2\upsilon)},$$

(15)

and

$$\mu = G = \frac{E}{2(1+\upsilon)},$$

(16)

where E is the complex compressional modulus (N/m^2), G is the complex shear modulus (N/m^2), and υ is the Poisson's ratio of the material (dimensionless).

The conditions of infinite length, two-dimensional response ($y = 0$), and steady-state response are now imposed, allowing the scalar and vector potential to be written as

$$\phi(x,z,t) = \Phi(z)\exp(ik_x x)\exp(i\omega t),$$

(17)

and

$$\psi_y(x,z,t) = \Psi(z)\exp(ik_x x)\exp(i\omega t),$$

(18)

where k_x is the wavenumber of excitation in the x-direction (rad/m), ω is the frequency of excitation (rad/s), and i is the square root of -1. Note that equations (17) and (18) are valid on every layer.

Inserting equation (17) into equation (11) yields

$$\frac{d^2\Phi(z)}{dz^2} + \alpha^2\Phi(z) = 0, \tag{19}$$

where

$$\alpha = \sqrt{k_d^2 - k_x^2}, \tag{20}$$

and

$$k_d = \frac{\omega}{c_d}. \tag{21}$$

Inserting equation (18) into equation (12) yields

$$\frac{d^2\Psi(z)}{dz^2} + \beta^2\Psi(z) = 0, \tag{22}$$

where

$$\beta = \sqrt{k_s^2 - k_x^2}, \tag{23}$$

and

$$k_s = \frac{\omega}{c_s}. \tag{24}$$

The solution to equation (19) is

$$\Phi(z) = A(k_x,\omega)\exp(i\alpha z) + B(k_x,\omega)\exp(-i\alpha z), \tag{25}$$

and the solution to equation (22) is

$$\Psi(z) = C(k_x,\omega)\exp(i\beta z) + D(k_x,\omega)\exp(-i\beta z), \tag{26}$$

where A, B, C, and D are constants that are determined below. The displacements can now be written as functions of the unknown constants. They are

$$u_z(x,z,t) = \{ \, i\alpha[A(k_x,\omega)\exp(i\alpha z) - B(k_x,\omega)\exp(-i\alpha z)] +$$
$$ik_x[C(k_x,\omega)\exp(i\beta z) + D(k_x,\omega)\exp(-i\beta z)] \,\}\exp(ik_x x)\exp(i\omega t), \tag{27}$$

and

$$u_x(x,z,t) = \{ \, ik_x[A(k_x,\omega)\exp(i\alpha z) + B(k_x,\omega)\exp(-i\alpha z)] -$$
$$i\beta[C(k_x,\omega)\exp(i\beta z) - D(k_x,\omega)\exp(-i\beta z)] \,\}\exp(ik_x x)\exp(i\omega t). \tag{28}$$

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The solution to the constants is determined by formulating the problem based on the number of plates and the presence or absence of a fluid load. This is done in sections 3 - 8. Four derivatives of equations (27) and (28), which are used in the solution of the constants, are

$$
\frac{\partial u_x(x,z,t)}{\partial x} = \{ \ -k_x^2 [A(k_x,\omega)\exp(i\alpha z) + B(k_x,\omega)\exp(-i\alpha z)] +
$$
$$
\beta k_x [C(k_x,\omega)\exp(i\beta z) - D(k_x,\omega)\exp(-i\beta z)] \} \exp(ik_x x)\exp(i\omega t), \qquad (29)
$$

$$
\frac{\partial u_x(x,z,t)}{\partial z} = \{ \ -k_x\alpha [A(k_x,\omega)\exp(i\alpha z) - B(k_x,\omega)\exp(-i\alpha z)] +
$$
$$
\beta^2 [C(k_x,\omega)\exp(i\beta z) + D(k_x,\omega)\exp(-i\beta z)] \} \exp(ik_x x)\exp(i\omega t), \qquad (30)
$$

$$
\frac{\partial u_z(x,z,t)}{\partial z} = \{ \ -\alpha^2 [A(k_x,\omega)\exp(i\alpha z) + B(k_x,\omega)\exp(-i\alpha z)] -
$$
$$
\beta k_x [C(k_x,\omega)\exp(i\beta z) - D(k_x,\omega)\exp(-i\beta z)] \} \exp(ik_x x)\exp(i\omega t), \qquad (31)
$$

and

$$
\frac{\partial u_z(x,z,t)}{\partial x} = \{ \ k_x\alpha [A(k_x,\omega)\exp(i\alpha z) - B(k_x,\omega)\exp(-i\alpha z)] -
$$
$$
k_x^2 [C(k_x,\omega)\exp(i\beta z) + D(k_x,\omega)\exp(-i\beta z)] \} \exp(ik_x x)\exp(i\omega t). \qquad (32)
$$

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3. SINGLE PATE WITH NO FLUID LOAD

The first transfer function derived is a sigle plate with no fluid load. The structural load consists of an applied pressure at definite wavenumber and frequency at location $z = b$, as shown in figure 2. The normal and tangential stresses in the system at the boundaries are

$$\tau_{zz}(x,b,t) = (\lambda + 2\mu)\frac{\partial u_z(x,b,t)}{\partial z} + \lambda\frac{\partial u_x(x,b,t)}{\partial x} = -p_s(x,b,t), \tag{33}$$

$$\tau_{zx}(x,b,t) = \mu\left[\frac{\partial u_x(x,b,t)}{\partial z} + \frac{\partial u_z(x,b,t)}{\partial x}\right] = 0, \tag{34}$$

$$\tau_{zz}(x,a,t) = (\lambda + 2\mu)\frac{\partial u_z(x,a,t)}{\partial z} + \lambda\frac{\partial u_x(x,a,t)}{\partial x} = 0, \tag{35}$$

and

$$\tau_{zx}(x,a,t) = \mu\left[\frac{\partial u_x(x,a,t)}{\partial z} + \frac{\partial u_z(x,a,t)}{\partial x}\right] = 0, \tag{36}$$

where $p_S(x,b,t)$ is the structural load applied at $z = b$. This load is modeled as a function at definite wavenumber and frequency as

$$p_s(x,b,t) = P_s \exp(ik_x x)\exp(i\omega t). \tag{37}$$

Combining equations (29) - (37) yields the four-by-four linear system of equations

$$\mathbf{Ax} = \mathbf{b}, \tag{38}$$

where the entries of equation (38) are

$$A_{1,1} = \left(-\alpha^2\lambda - 2\alpha^2\mu - \lambda k_x^2\right)\exp(i\alpha b), \tag{39}$$

$$A_{1,2} = \left(-\alpha^2\lambda - 2\alpha^2\mu - \lambda k_x^2\right)\exp(-i\alpha b), \tag{40}$$

$$A_{1,3} = \left(-2k_x\beta\mu\right)\exp(i\beta b), \tag{41}$$

$$A_{1,4} = \left(2k_x\beta\mu\right)\exp(-i\beta b), \tag{42}$$

$$A_{2,1} = \left(-2\mu k_x\alpha\right)\exp(i\alpha b), \tag{43}$$

$$A_{2,2} = \left(2\mu k_x\alpha\right)\exp(-i\alpha b), \tag{44}$$
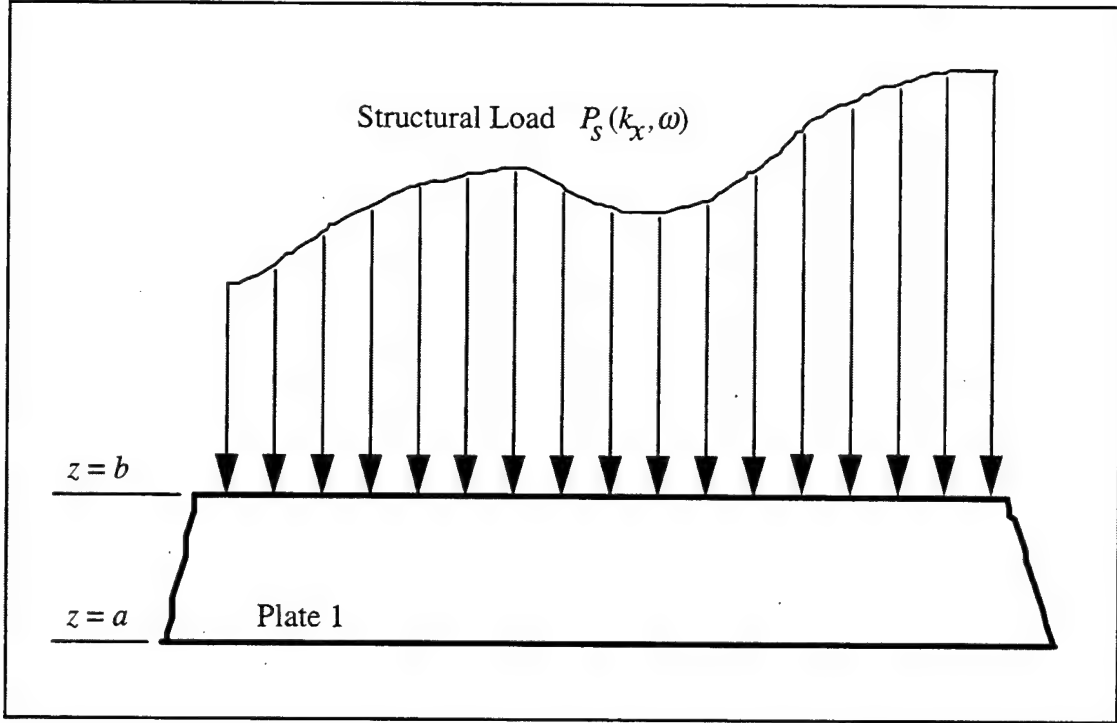
Figure 2. Single Plate with No Fluid Load

$$A_{2,3} = \left(\mu\beta^2 - \mu k_x^2\right)\exp(i\beta b), \tag{45}$$

$$A_{2,4} = \left(\mu\beta^2 - \mu k_x^2\right)\exp(-i\beta b), \tag{46}$$

$$A_{3,1} = \left(-\alpha^2\lambda - 2\alpha^2\mu - \lambda k_x^2\right)\exp(i\alpha a), \tag{47}$$

$$A_{3,2} = \left(-\alpha^2\lambda - 2\alpha^2\mu - \lambda k_x^2\right)\exp(-i\alpha a), \tag{48}$$

$$A_{3,3} = \left(-2k_x\beta\mu\right)\exp(i\beta a), \tag{49}$$

$$A_{3,4} = \left(2k_x\beta\mu\right)\exp(-i\beta a), \tag{50}$$

$$A_{4,1} = \left(-2\mu k_x\alpha\right)\exp(i\alpha a), \tag{51}$$

$$A_{4,2} = \left(2\mu k_x\alpha\right)\exp(-i\alpha a), \tag{52}$$

$$A_{4,3} = \left(\mu\beta^2 - \mu k_x^2\right)\exp(i\beta a), \tag{53}$$

$$A_{4,4} = \left(\mu\beta^2 - \mu k_x^2\right)\exp(-i\beta a), \tag{54}$$

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$$x_{1,1} = A(k_x, \omega), \tag{55}$$

$$x_{2,1} = B(k_x, \omega), \tag{56}$$

$$x_{3,1} = C(k_x, \omega), \tag{57}$$

$$x_{4,1} = D(k_x, \omega), \tag{58}$$

$$b_{1,1} = -P_s(k_x, \omega), \tag{59}$$

$$b_{2,1} = 0, \tag{60}$$

$$b_{3,1} = 0, \tag{61}$$

and

$$b_{4,1} = 0. \tag{62}$$

Using equations (39) - (62), the solution to the constants A, B, C, and D can be found by

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}. \tag{63}$$

Additionally, the transfer function between the normal displacement at location z and the structural load can be written as

$$T(k_x, \omega) = \frac{U_z(z, k_x, \omega)}{P_s(k_x, \omega)} = A(k_x, \omega)i\alpha\exp(i\alpha z) - B(k_x, \omega)i\alpha\exp(-i\alpha z) +$$

$$C(k_x, \omega)ik_x\exp(i\beta z) + D(k_x, \omega)ik_x\exp(-i\beta z), \tag{64}$$

or the normal stress and the structural load

$$T(k_x, \omega) = \frac{T_{zz}(z, k_x, \omega)}{P_s(k_x, \omega)} = A(k_x, \omega)\left(-\alpha^2\lambda - 2\alpha^2\mu - \lambda k_x^2\right)\exp(i\alpha z) +$$

$$B(k_x, \omega)\left(-\alpha^2\lambda - 2\alpha^2\mu - \lambda k_x^2\right)\exp(-i\alpha z) +$$

$$C(k_x, \omega)(-2k_x\beta\mu)\exp(i\beta z) + D(k_x, \omega)(2k_x\beta\mu)\exp(-i\beta z). \tag{65}$$

The transfer function in equation (64) can be compared to another transfer function derived using the thin plate theory to ensure that it is accurate. The thin plate equation of motion is

$$\hat{B}\frac{\partial^4 u_z(x,t)}{\partial x^4} + \rho h\frac{\partial^2 u_z(x,t)}{\partial t^2} = -p_s(x,t), \tag{66}$$

where $\hat{B}$ is the bending stress expressed as

$$\hat{B} = \frac{Eh^3}{12(1-v^2)}, \tag{67}$$

and h is the thickness of the plate (m) and corresponds to b-a in figure 2. The transfer function between transverse displacement and applied force for the thin plate is

$$T(k_x,\omega) = \frac{U_z(k_x,\omega)}{P_s(k_x,\omega)} = \frac{-1}{\left(\hat{B}k_x^4 - \rho h\omega^2\right)}. \tag{68}$$

The flexural wavenumber of this system is at

$$k_f = \left[\frac{\rho h\omega^2}{\hat{B}}\right]^{1/4}, \tag{69}$$

where k_f has units of rad/m. At this wavenumber, the response of the system will be extremely large compared to its response at other wavenumbers.

Figure 3 is a plot of displacement in the z-direction (normal) divided by normal pressure versus wavenumber in the x-direction at a forcing frequency of 500 Hz evaluated at $z = b$. The solid line represents the thick plate theory (equation (64)) and the x's represent the thin plate theory (equation (68)). The flexural wavenumber calculated from equation (69) is denoted on the plot. In this example, the thickness of the plate is very small ($h = 0.01$ m) so that a direct comparison between the thick and thin plate theories can be made. Other parameters used to formulate this model are: bottom of the plate (a) is -0.01 m, top of the plate (b) is 0 m, Young's modulus (E) is 13.2e9 (1+0.03i) N/m^2, Poisson's ratio (v) is 0.30, and density (ρ) is 1938 kg/m^3. Only positive values of wavenumber are shown, as the function is symmetric about $k = 0$. Note that there is almost complete agreement between the thin and thick plate theories for this specific example. Figure 4 is a plot of displacement in the z-direction (normal) divided by normal pressure versus wavenumber in the x-direction at a forcing frequency of 500 Hz and a plate thickness of $h = 0.1$ m evaluated at $z = b$. The two models begin to diverge as the wavenumber increases. This is due to the inclusion of the rotary inertia terms in the thick plate model that are not included in the thin plate model. Figure 5 is a plot of the wavenumber-frequency ($k\omega$) surface of the system using the thick plate equations of motion (equation (64)) and the parameters that were used to formulate figure 4. The color scale to the right of the plot is in decibels.

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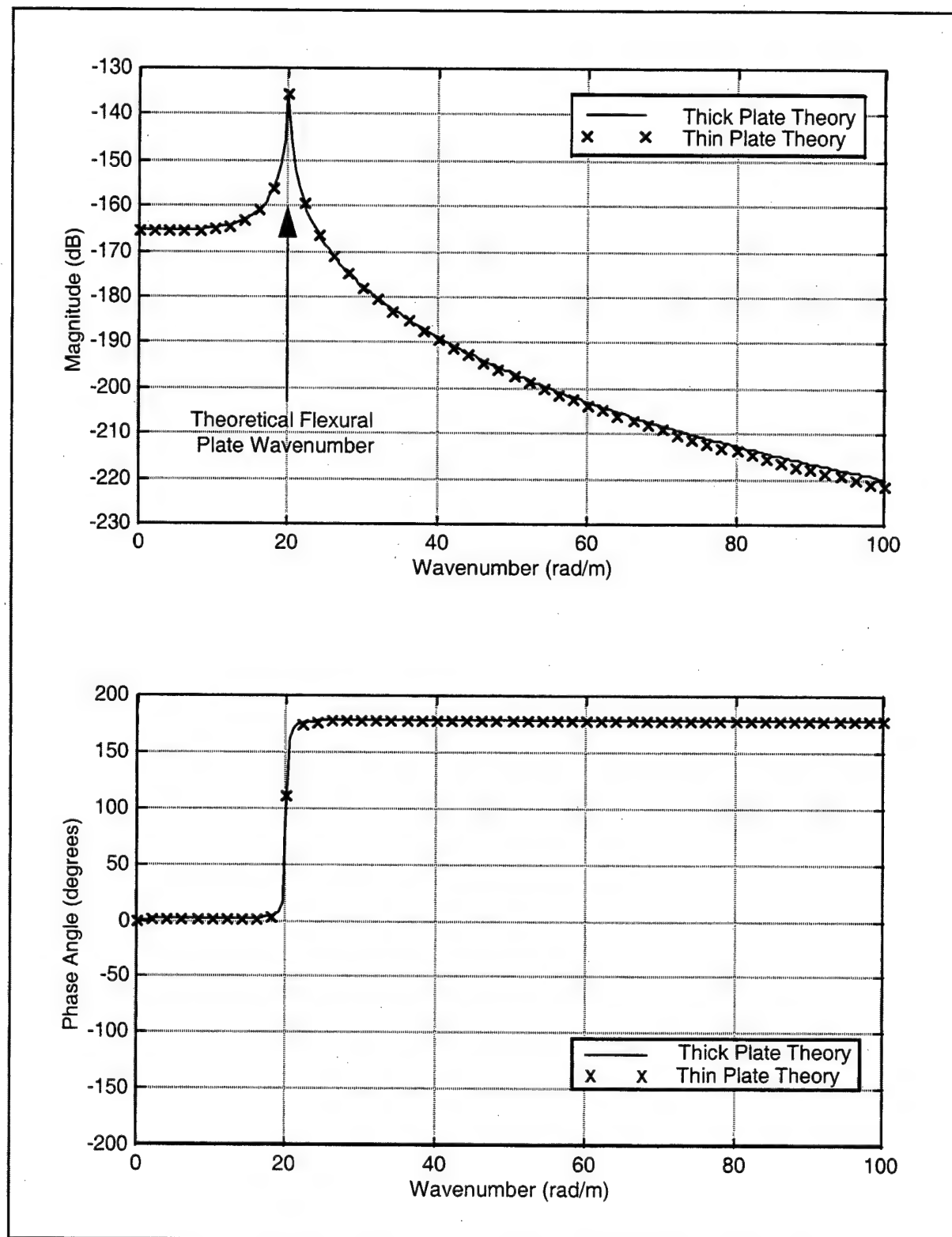
Figure 3. Transfer Function of a Single Plate with No Fluid Load (Equations (64) and (68)) at 500 Hz with h = 0.01 m

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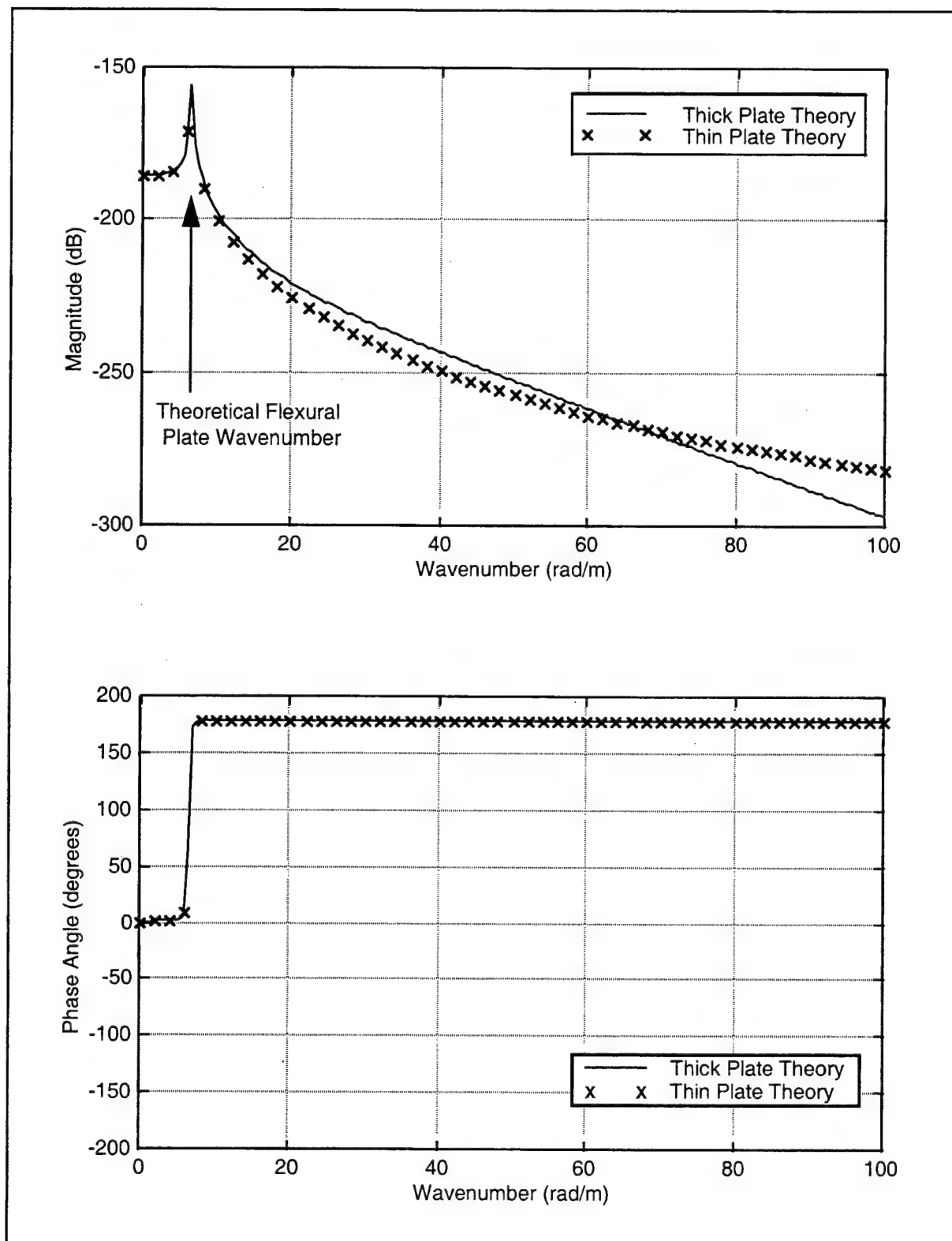
Figure 4. Transfer Function of a Single Plate with No Fluid Load (Equations (64) and (68)) at 500 Hz with h = 0.1 m
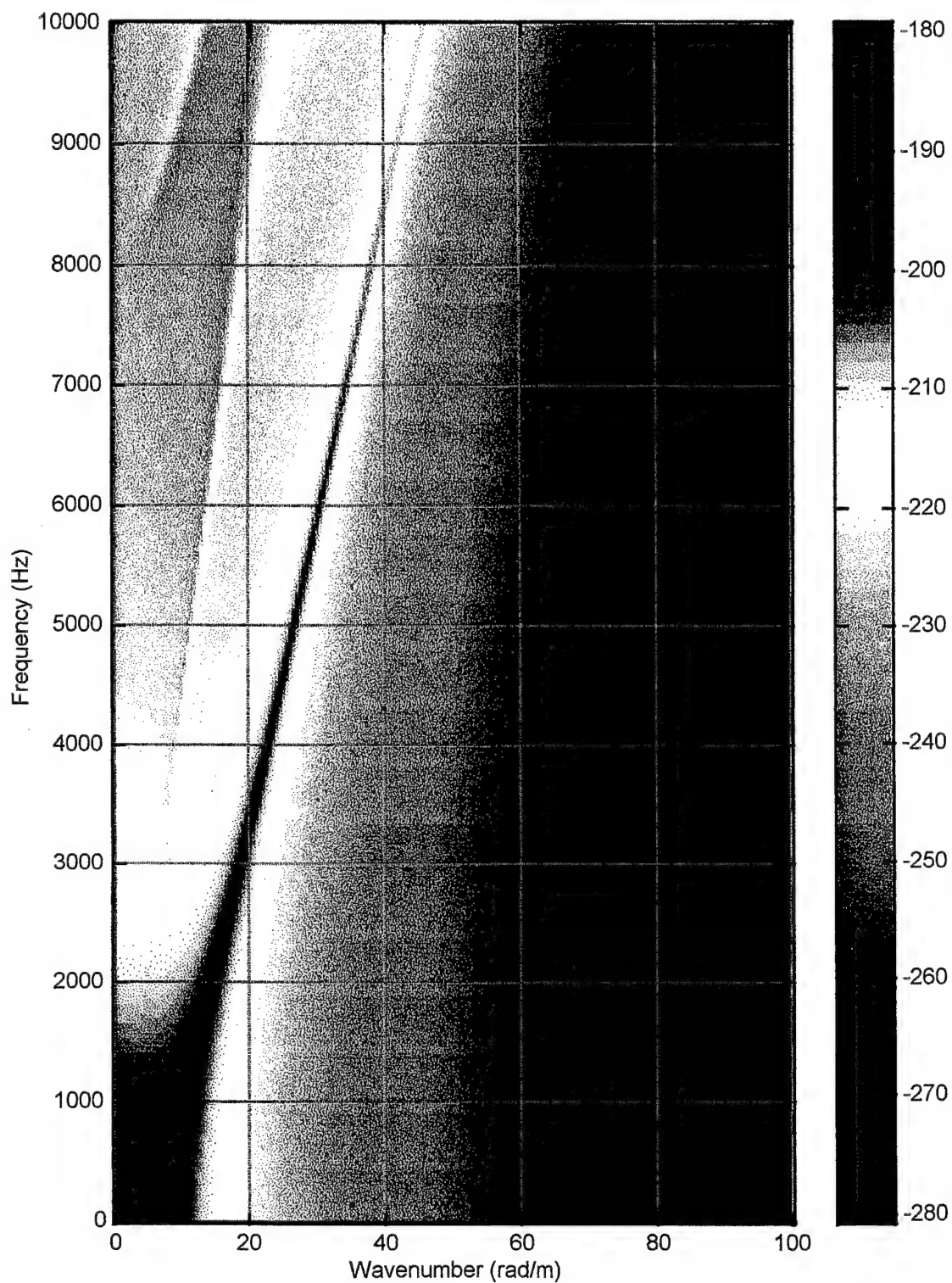
Figure 5. Transfer Surface of a Single Plate with No Fluid Load (Equation (64)) with h = 0.1 m

4. SINGLE PLATE WITH A FLUID LOAD

The second transfer function derived is a single plate with a fluid load on one side. The applied load is modeled as an incident pressure wave in the fluid at definite wavenumber and frequency at location $z = b$, as shown in figure 6. The normal and tangential stresses in the system at the boundaries are

$$\tau_{zz}(x,b,t) = (\lambda + 2\mu)\frac{\partial u_z(x,b,t)}{\partial z} + \lambda\frac{\partial u_x(x,b,t)}{\partial x} = -p_a(x,b,t), \tag{70}$$

$$\tau_{zx}(x,b,t) = \mu\left[\frac{\partial u_x(x,b,t)}{\partial z} + \frac{\partial u_z(x,b,t)}{\partial x}\right] = 0, \tag{71}$$

$$\tau_{zz}(x,a,t) = (\lambda + 2\mu)\frac{\partial u_z(x,a,t)}{\partial z} + \lambda\frac{\partial u_x(x,a,t)}{\partial x} = 0, \tag{72}$$

and

$$\tau_{zx}(x,a,t) = \mu\left[\frac{\partial u_x(x,a,t)}{\partial z} + \frac{\partial u_z(x,a,t)}{\partial x}\right] = 0, \tag{73}$$

where $p_a(x,b,t)$ represents the (acoustic) pressure of the fluid load on the plate and includes the applied load.

The acoustic pressure in the outer fluid is governed by the wave equation and is written in Cartesian coordinates (reference 4) as

$$\frac{\partial^2 p_a(x,z,t)}{\partial z^2} + \frac{\partial^2 p_a(x,z,t)}{\partial x^2} - \frac{1}{c_f^2}\frac{\partial^2 p_a(x,z,t)}{\partial t^2} = 0, \tag{74}$$

where $p_a(x,z,t)$ is the pressure (N/m^2), z is the spatial location (m) normal to the plate, and c_f is the compressional wavespeed of the fluid (m/s). The acoustic pressure is modeled as a function at definite wavenumber and frequency as

$$p_a(x,z,t) = P_a(z,k_x,\omega)\exp(ik_x x)\exp(i\omega t). \tag{75}$$

Inserting equation (75) into (74) and solving the resulting ordinary differential equation yields

$$P_a(z,k_x,\omega) = G(k_x,\omega)\exp(i\gamma z) + P_I(k_x,\omega)\exp(-i\gamma z), \tag{76}$$

where the first term on the right-hand side represents the reradiated pressure field and the second term represents the applied incident pressure field (the forcing function) acting on the structure. In equation (76),
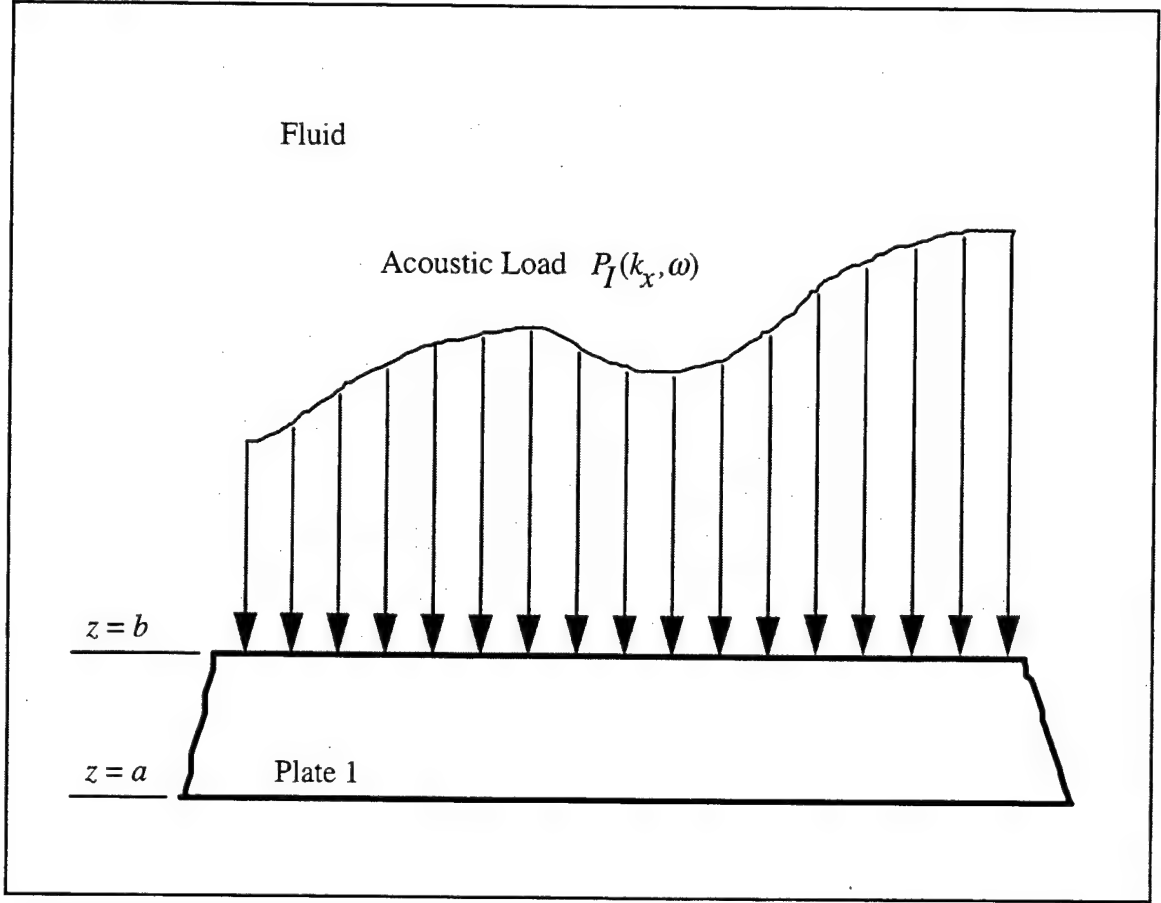
Figure 6. Single Plate with Fluid Load

$$\gamma = \sqrt{\left(\frac{\omega}{c_f}\right)^2 - k_x^2} \; , \tag{77}$$

where γ is purely real or imaginary, depending on the sign of the argument. When the sign of the argument is positive, the analysis is in the acoustic cone; when the sign of the argument is negative, the analysis is in the nonacoustic region. The interface between the fluid and solid surface of the plate satisfies the linear momentum equation (reference 5), which is

$$\rho_f \frac{\partial^2 u_z(x,b,t)}{\partial t^2} = -\frac{\partial p_a(x,b,t)}{\partial z} \; , \tag{78}$$

where ρ_f is the density of the fluid (kg/m^3).

Combining equations (70) - (78) yields the four-by-four linear system of equations

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$$\mathbf{Ax = b} \,, \tag{79}$$

where the entries of equation (79) are

$$A_{1,1} = \left(-\alpha^2\lambda - 2\alpha^2\mu - \lambda k_x^2 + \frac{\rho_f \omega^2 \alpha}{\gamma} \right) \exp(i\alpha b) \,, \tag{80}$$

$$A_{1,2} = \left(-\alpha^2\lambda - 2\alpha^2\mu - \lambda k_x^2 - \frac{\rho_f \omega^2 \alpha}{\gamma} \right) \exp(-i\alpha b) \,, \tag{81}$$

$$A_{1,3} = \left(-2 k_x \beta \mu + \frac{\rho_f \omega^2 k_x}{\gamma} \right) \exp(i\beta b) \,, \tag{82}$$

$$A_{1,4} = \left(2 k_x \beta \mu + \frac{\rho_f \omega^2 k_x}{\gamma} \right) \exp(-i\beta b) \,, \tag{83}$$

$$A_{2,1} = \left(-2\mu k_x \alpha \right) \exp(i\alpha b) \,, \tag{84}$$

$$A_{2,2} = \left(2\mu k_x \alpha \right) \exp(-i\alpha b) \,, \tag{85}$$

$$A_{2,3} = \left(\mu \beta^2 - \mu k_x^2 \right) \exp(i\beta b) \,, \tag{86}$$

$$A_{2,4} = \left(\mu \beta^2 - \mu k_x^2 \right) \exp(-i\beta b) \,, \tag{87}$$

$$A_{3,1} = \left(-\alpha^2\lambda - 2\alpha^2\mu - \lambda k_x^2 \right) \exp(i\alpha a) \,, \tag{88}$$

$$A_{3,2} = \left(-\alpha^2\lambda - 2\alpha^2\mu - \lambda k_x^2 \right) \exp(-i\alpha a) \,, \tag{89}$$

$$A_{3,3} = \left(-2 k_x \beta \mu \right) \exp(i\beta a) \,, \tag{90}$$

$$A_{3,4} = \left(2 k_x \beta \mu \right) \exp(-i\beta a) \,, \tag{91}$$

$$A_{4,1} = \left(-2\mu k_x \alpha \right) \exp(i\alpha a) \,, \tag{92}$$

$$A_{4,2} = \left(2\mu k_x \alpha \right) \exp(-i\alpha a) \,, \tag{93}$$

$$A_{4,3} = \left(\mu \beta^2 - \mu k_x^2 \right) \exp(i\beta a) \,, \tag{94}$$

$$A_{4,4} = \left(\mu\beta^2 - \mu k_x^2\right)\exp(-i\beta a),$$
(95)

$$x_{1,1} = A(k_x,\omega),$$
(96)

$$x_{2,1} = B(k_x,\omega),$$
(97)

$$x_{3,1} = C(k_x,\omega),$$
(98)

$$x_{4,1} = D(k_x,\omega),$$
(99)

$$b_{1,1} = -2P_I(k_x,\omega)\exp(-i\gamma b),$$
(100)

$$b_{2,1} = 0,$$
(101)

$$b_{3,1} = 0,$$
(102)

and

$$b_{4,1} = 0.$$
(103)

Using equations (80) - (103), the solution to the constants A, B, C, and D can be found by

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}.$$
(104)

Additionally, the transfer function between the normal displacement at location z and the acoustic load can be written as

$$T(k_x,\omega) = \frac{U_z(z,k_x,\omega)}{P_I(k_x,\omega)\exp(-i\gamma b)} = A(k_x,\omega)i\alpha\exp(i\alpha z) - B(k_x,\omega)i\alpha\exp(-i\alpha z) +$$

$$C(k_x,\omega)ik_x\exp(i\beta z) + D(k_x,\omega)ik_x\exp(-i\beta z),$$
(105)

or the normal stress and the structural load

$$T(k_x,\omega) = \frac{T_{zz}(z,k_x,\omega)}{P_I(k_x,\omega)\exp(-i\gamma b)} = A(k_x,\omega)\left(-\alpha^2\lambda - 2\alpha^2\mu - \lambda k_x^2\right)\exp(i\alpha z) +$$

$$B(k_x,\omega)\left(-\alpha^2\lambda - 2\alpha^2\mu - \lambda k_x^2\right)\exp(-i\alpha z) +$$

$$C(k_x,\omega)(-2k_x\beta\mu)\exp(i\beta z) + D(k_x,\omega)(2k_x\beta\mu)\exp(-i\beta z).$$
(106)

The transfer function in equation (105) can be compared to another transfer function derived using the thin plate theory to ensure that it is accurate. This transfer function is derived by equating equation (66) to equation (76) and applying equation (78), which yields

$$T(k_x,\omega) = \frac{U_z(k_x,\omega)}{P_I(k_x,\omega)\exp(-i\gamma b)} = \frac{-2}{\left(\hat{B}k_x^4 - \rho h\omega^2 + \dfrac{\rho_f\omega^2}{i\gamma}\right)} . \qquad (107)$$

The flexural wavenumber of the fluid loaded plate system is at

$$k_f' = k_f\left[1 + \frac{\rho_f}{h\rho k_f}\right]^{1/4} . \qquad (108)$$

where k_f' has units of rad/m and k_f is determined using equation (69). At this wavenumber, the response of the system will be extremely large compared to its response at other wavenumbers.

Figure 7 is a plot of displacement in the z-direction (normal) divided by normal pressure versus wavenumber in the x-direction at a forcing frequency of 500 Hz evaluated at $z = b$. The solid line represents the thick plate theory with a fluid load (equation (105)) and the x's represent the thin plate theory with a fluid load (equation (107)). The fluid-loaded flexural wavenumber calculated from equation (108) is denoted on the plot. In this example, the thickness of the plate is very small ($h = 0.01$ m) so that a direct comparison between the thick and thin plate theories can be made. Other parameters used to formulate this model are: bottom of the plate (a) is -0.01 m, top of the plate (b) is 0 m, Young's modulus (E) is 13.2e9 (1+0.03i) N/m^2, Poisson's ratio (v) is 0.30, density of the plate (ρ) is 1938 kg/m^3, compressional wavespeed of the fluid (c_f) is 1500 m/s, and density of the fluid (ρ_f) is 1025 kg/m^3. Only positive values of wavenumber are shown, as the function is symmetric about $k = 0$. Note that there is almost complete agreement between the thin and thick plate theories for this specific example. Figure 8 is a plot of displacement in the z-direction (normal) divided by normal pressure versus wavenumber in the x-direction at a forcing frequency of 500 Hz and a plate thickness of $h = 0.1$ m evaluated at $z = b$. The two models begin to diverge as wavenumber increases. This is due to the inclusion of the rotary inertia terms in the thick plate model which are not included in the thin plate model. Figure 9 is a plot of the wavenumber-frequency ($k\omega$) surface of the system using the thick plate equations of motion (equation (105)) and the parameters that were used to formulate figure 7. The color scale to the right of the plot is in decibels.
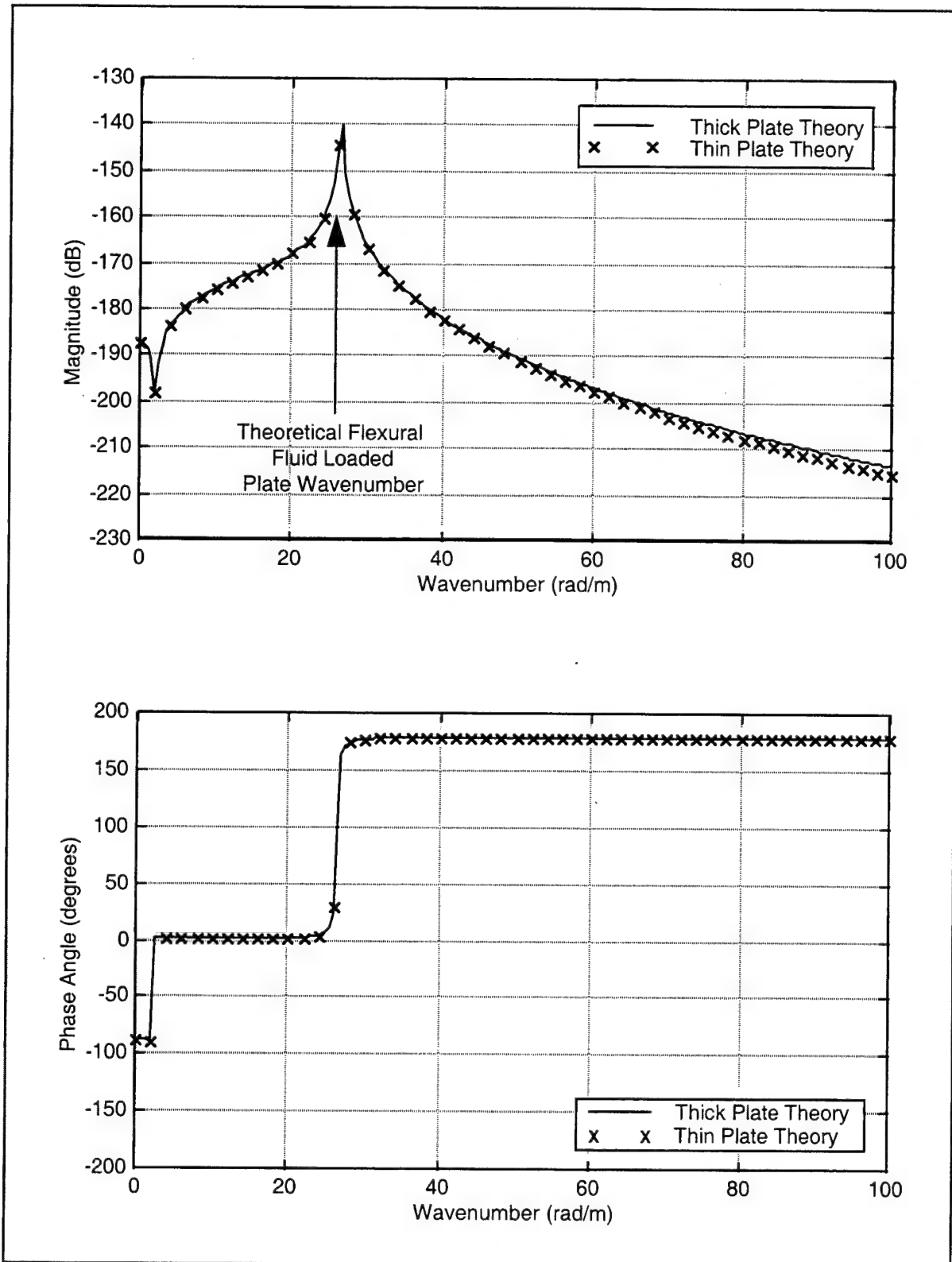
Figure 7. Transfer Function of a Single Plate with Fluid Load (Equations (105) and (107)) at 500 Hz with h = 0.01 m
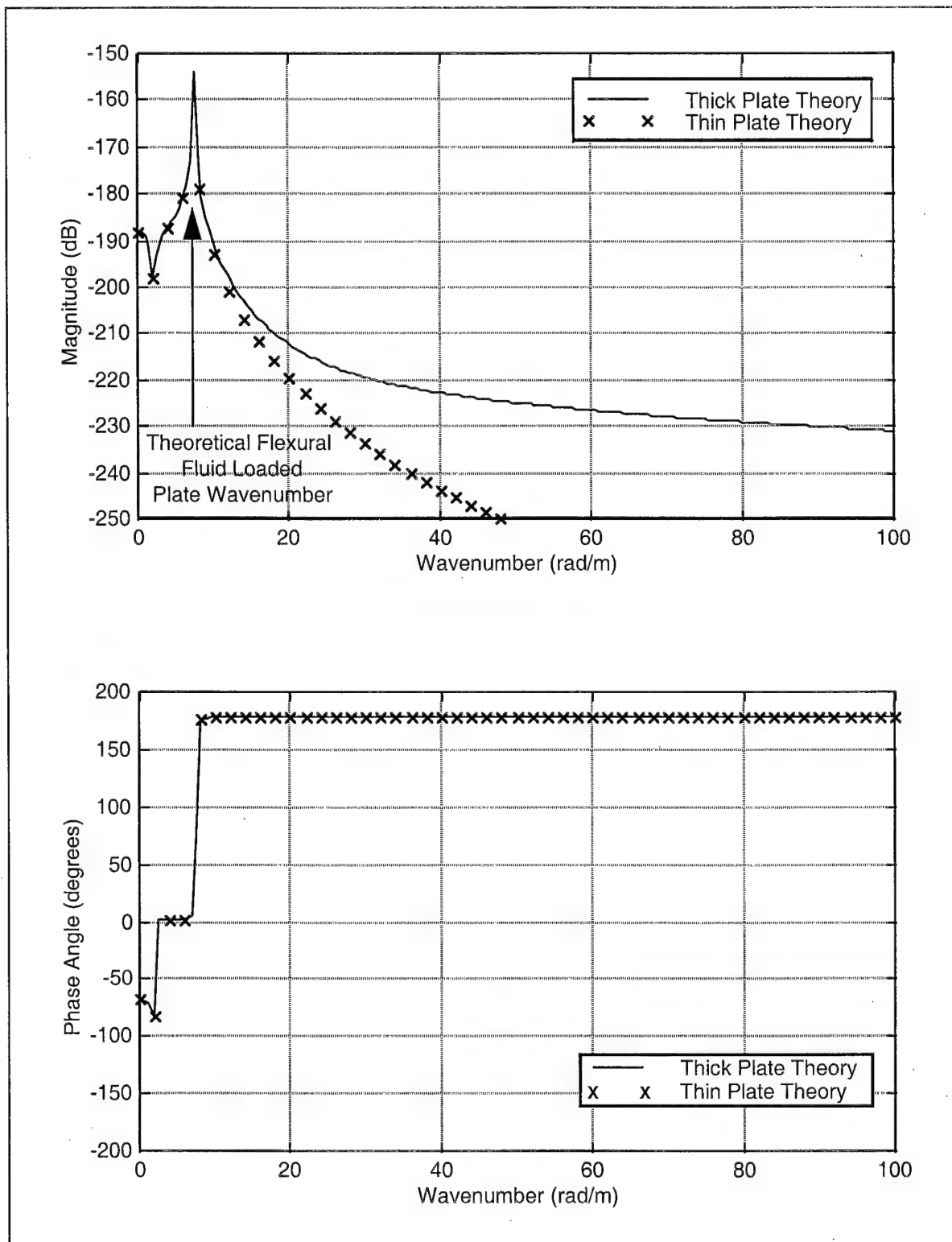
*Figure 8. Transfer Function of a Single Plate with Fluid Load
(Equations (105) and (107)) at 500 Hz with h = 0.1 m*

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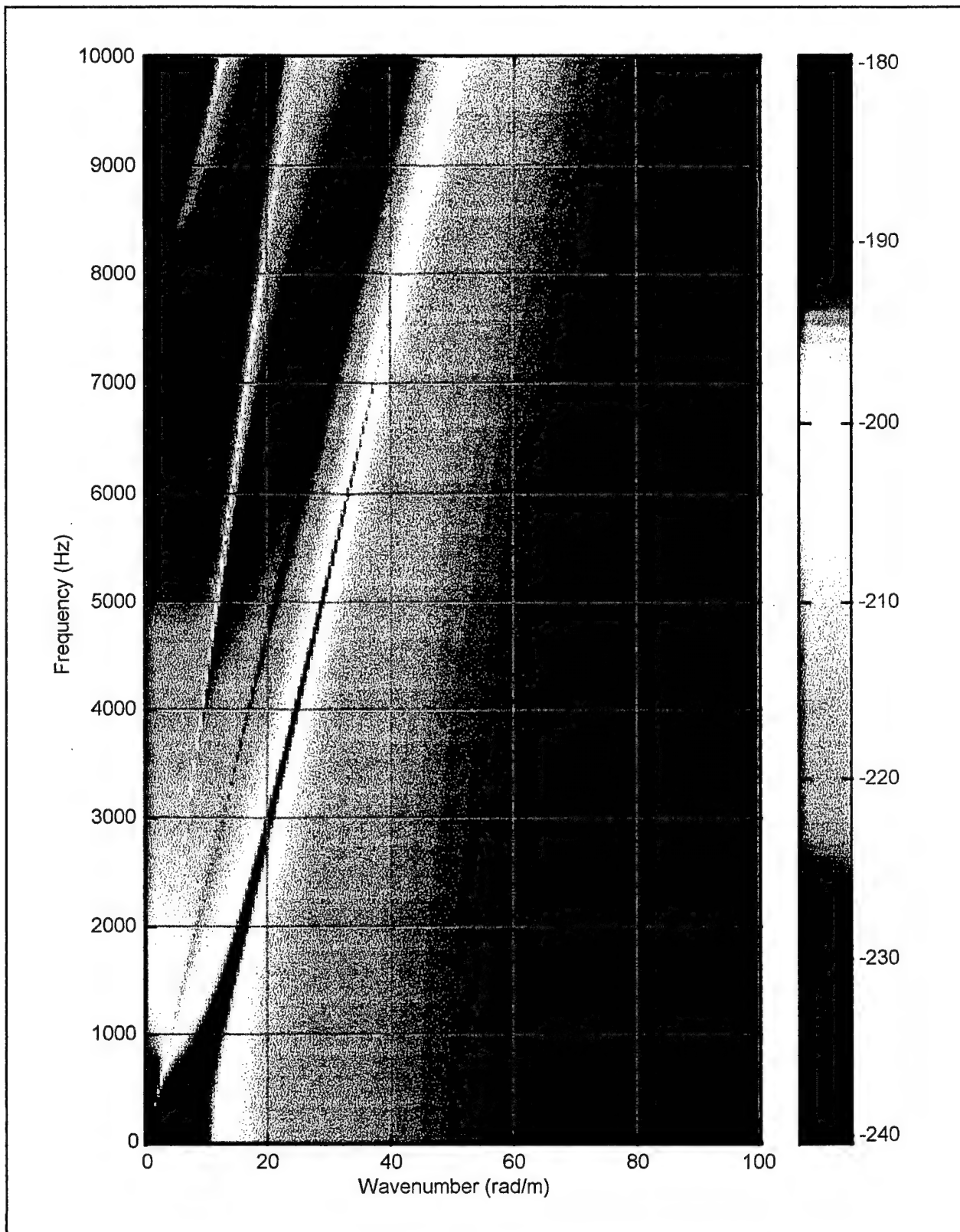
Figure 9. Transfer Surface of a Single Plate with Fluid Load
(Equation (105)) with h = 0.1 m

5. DOUBLE PLATE WITH NO FLUID LOAD

The third transfer function derived is a double plate with no fluid load. The structural load consists of an applied pressure at definite wavenumber and frequency at location $z = c$, as shown in figure 10. The normal and tangential stresses in the system at the boundary $z = c$ are

$$\tau_{zz}(x,c,t) = (\lambda_2 + 2\mu_2)\frac{\partial u_z(x,c,t)}{\partial z} + \lambda_2 \frac{\partial u_x(x,c,t)}{\partial x} = -p_s(x,c,t),$$ (109)

and

$$\tau_{zx}(x,c,t) = \mu_2\left[\frac{\partial u_x(x,c,t)}{\partial z} + \frac{\partial u_z(x,c,t)}{\partial x}\right] = 0,$$ (110)

where the subscript 2 denotes plate 2. The interface between plates 2 and 1 requires four equations. The first two are displacement constraints, which are

$$u_z(x,b,t)\big|_{\text{plate 2}} = u_z(x,b,t)\big|_{\text{plate1}},$$ (111)

and

$$u_x(x,b,t)\big|_{\text{plate 2}} = u_x(x,b,t)\big|_{\text{plate1}}.$$ (112)

The second two are stress constraints, which are

$$\tau_{zz}(x,b,t)\big|_{\text{plate 2}} = \tau_{zz}(x,b,t)\big|_{\text{plate1}},$$ (113)

and

$$\tau_{zx}(x,b,t)\big|_{\text{plate 2}} = \tau_{zx}(x,b,t)\big|_{\text{plate1}}.$$ (114)

Finally, the normal and tangential stresses in the system at the boundary $z = a$ are

$$\tau_{zz}(x,a,t) = (\lambda_1 + 2\mu_1)\frac{\partial u_z(x,a,t)}{\partial z} + \lambda_1 \frac{\partial u_x(x,a,t)}{\partial x} = 0,$$ (115)

and

$$\tau_{zx}(x,a,t) = \mu_1\left[\frac{\partial u_x(x,a,t)}{\partial z} + \frac{\partial u_z(x,a,t)}{\partial x}\right] = 0,$$ (116)

where the subscript 1 denotes plate 1.
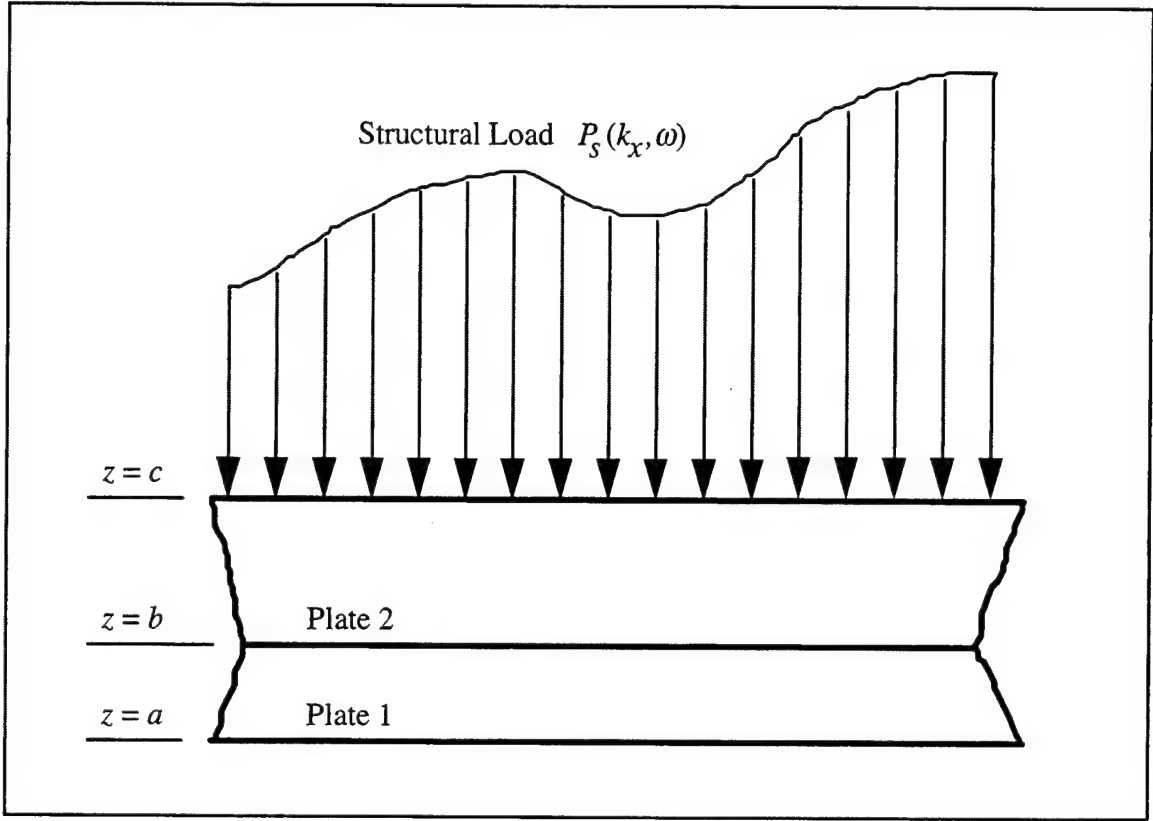
Figure 10. Double Plate with No Fluid Load

Combining equations (109) - (116) yields the eight-by-eight linear system of equations

$$\mathbf{Ax} = \mathbf{b},\tag{117}$$

where the entries of equation (117) are

$$A_{1,1} = \left(-\alpha_2^2 \lambda_2 - 2\alpha_2^2 \mu_2 - \lambda_2 k_x^2\right)\exp(i\alpha_2 c),\tag{118}$$

$$A_{1,2} = \left(-\alpha_2^2 \lambda_2 - 2\alpha_2^2 \mu_2 - \lambda_2 k_x^2\right)\exp(-i\alpha_2 c),\tag{119}$$

$$A_{1,3} = \left(-2k_x \beta_2 \mu_2\right)\exp(i\beta_2 c),\tag{120}$$

$$A_{1,4} = \left(2k_x \beta_2 \mu_2\right)\exp(-i\beta_2 c),\tag{121}$$

$$A_{1,5} = 0,\tag{122}$$

$$A_{1,6} = 0,\tag{123}$$

$$A_{1,7} = 0,$$
(124)

$$A_{1,8} = 0,$$
(125)

$$A_{2,1} = \left(-2\mu_2 k_x \alpha_2\right)\exp(i\alpha_2 c),$$
(126)

$$A_{2,2} = \left(2\mu_2 k_x \alpha_2\right)\exp(-i\alpha_2 c),$$
(127)

$$A_{2,3} = \left(\mu_2 \beta_2^2 - \mu_2 k_x^2\right)\exp(i\beta_2 c),$$
(128)

$$A_{2,4} = \left(\mu_2 \beta_2^2 - \mu_2 k_x^2\right)\exp(-i\beta_2 c),$$
(129)

$$A_{2,5} = 0,$$
(130)

$$A_{2,6} = 0,$$
(131)

$$A_{2,7} = 0,$$
(132)

$$A_{2,8} = 0,$$
(133)

$$A_{3,1} = \left(i\alpha_2\right)\exp(i\alpha_2 b),$$
(134)

$$A_{3,2} = \left(-i\alpha_2\right)\exp(-i\alpha_2 b),$$
(135)

$$A_{3,3} = \left(i k_x\right)\exp(i\beta_2 b),$$
(136)

$$A_{3,4} = \left(i k_x\right)\exp(-i\beta_2 b),$$
(137)

$$A_{3,5} = \left(-i\alpha_1\right)\exp(i\alpha_1 b),$$
(138)

$$A_{3,6} = \left(i\alpha_1\right)\exp(-i\alpha_1 b),$$
(139)

$$A_{3,7} = \left(-i k_x\right)\exp(i\beta_1 b),$$
(140)

$$A_{3,8} = \left(-i k_x\right)\exp(-i\beta_1 b),$$
(141)

$$A_{4,1} = \left(i k_x\right)\exp(i\alpha_2 b),$$
(142)

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$$A_{4,2} = (ik_x)\exp(-i\alpha_2 b), \tag{143}$$

$$A_{4,3} = (-i\beta_2)\exp(i\beta_2 b), \tag{144}$$

$$A_{4,4} = (i\beta_2)\exp(-i\beta_2 b), \tag{145}$$

$$A_{4,5} = (-ik_x)\exp(i\alpha_1 b), \tag{146}$$

$$A_{4,6} = (-ik_x)\exp(-i\alpha_1 b), \tag{147}$$

$$A_{4,7} = (i\beta_1)\exp(i\beta_1 b), \tag{148}$$

$$A_{4,8} = (-i\beta_1)\exp(-i\beta_1 b), \tag{149}$$

$$A_{5,1} = \left(-\alpha_2^2\lambda_2 - 2\alpha_2^2\mu_2 - \lambda_2 k_x^2\right)\exp(i\alpha_2 b), \tag{150}$$

$$A_{5,2} = \left(-\alpha_2^2\lambda_2 - 2\alpha_2^2\mu_2 - \lambda_2 k_x^2\right)\exp(-i\alpha_2 b), \tag{151}$$

$$A_{5,3} = \left(-2k_x\beta_2\mu_2\right)\exp(i\beta_2 b), \tag{152}$$

$$A_{5,4} = \left(2k_x\beta_2\mu_2\right)\exp(-i\beta_2 b), \tag{153}$$

$$A_{5,5} = \left(\alpha_1^2\lambda_1 + 2\alpha_1^2\mu_1 + \lambda_1 k_x^2\right)\exp(i\alpha_1 b), \tag{154}$$

$$A_{5,6} = \left(\alpha_1^2\lambda_1 + 2\alpha_1^2\mu_1 + \lambda_1 k_x^2\right)\exp(-i\alpha_1 b), \tag{155}$$

$$A_{5,7} = \left(2k_x\beta_1\mu_1\right)\exp(i\beta_1 b), \tag{156}$$

$$A_{5,8} = \left(-2k_x\beta_1\mu_1\right)\exp(-i\beta_1 b), \tag{157}$$

$$A_{6,1} = \left(-2\mu_2 k_x\alpha_2\right)\exp(i\alpha_2 b), \tag{158}$$

$$A_{6,2} = \left(2\mu_2 k_x\alpha_2\right)\exp(-i\alpha_2 b), \tag{159}$$

$$A_{6,3} = \left(\mu_2\beta_2^2 - \mu_2 k_x^2\right)\exp(i\beta_2 b), \tag{160}$$

$$A_{6,4} = \left(\mu_2\beta_2^2 - \mu_2 k_x^2\right)\exp(-i\beta_2 b), \tag{161}$$

$$A_{6,5} = \left(2\mu_1 k_x\alpha_1\right)\exp(i\alpha_1 b), \tag{162}$$

$$A_{6,6} = \left(-2\mu_1 k_x \alpha_1\right)\exp(-\mathrm{i}\alpha_1 b), \tag{163}$$

$$A_{6,7} = \left(-\mu_1 \beta_1^2 + \mu_1 k_x^2\right)\exp(\mathrm{i}\beta_1 b), \tag{164}$$

$$A_{6,8} = \left(-\mu_1 \beta_1^2 + \mu_1 k_x^2\right)\exp(-\mathrm{i}\beta_1 b), \tag{165}$$

$$A_{7,1} = 0, \tag{166}$$

$$A_{7,2} = 0, \tag{167}$$

$$A_{7,3} = 0, \tag{168}$$

$$A_{7,4} = 0, \tag{169}$$

$$A_{7,5} = \left(-\alpha_1^2 \lambda_1 - 2\alpha_1^2 \mu_1 - \lambda_1 k_x^2\right)\exp(\mathrm{i}\alpha_1 a), \tag{170}$$

$$A_{7,6} = \left(-\alpha_1^2 \lambda_1 - 2\alpha_1^2 \mu_1 - \lambda_1 k_x^2\right)\exp(-\mathrm{i}\alpha_1 a), \tag{171}$$

$$A_{7,7} = \left(-2k_x \beta_1 \mu_1\right)\exp(\mathrm{i}\beta_1 a), \tag{172}$$

$$A_{7,8} = \left(2k_x \beta_1 \mu_1\right)\exp(-\mathrm{i}\beta_1 a), \tag{173}$$

$$A_{8,1} = 0, \tag{174}$$

$$A_{8,2} = 0, \tag{175}$$

$$A_{8,3} = 0, \tag{176}$$

$$A_{8,4} = 0, \tag{177}$$

$$A_{8,5} = \left(-2\mu_1 k_x \alpha_1\right)\exp(\mathrm{i}\alpha_1 a), \tag{178}$$

$$A_{8,6} = \left(2\mu_1 k_x \alpha_1\right)\exp(-\mathrm{i}\alpha_1 a), \tag{179}$$

$$A_{8,7} = \left(\mu_1 \beta_1^2 - \mu_1 k_x^2\right)\exp(\mathrm{i}\beta_1 a), \tag{180}$$

$$A_{8,8} = \left(\mu_1 \beta_1^2 - \mu_1 k_x^2\right)\exp(-\mathrm{i}\beta_1 a), \tag{181}$$

$$x_{1,1} = A_2(k_x, \omega),$$ (182)

$$x_{2,1} = B_2(k_x, \omega),$$ (183)

$$x_{3,1} = C_2(k_x, \omega),$$ (184)

$$x_{4,1} = D_2(k_x, \omega),$$ (185)

$$x_{5,1} = A_1(k_x, \omega),$$ (186)

$$x_{6,1} = B_1(k_x, \omega),$$ (187)

$$x_{7,1} = C_1(k_x, \omega),$$ (188)

$$x_{8,1} = D_1(k_x, \omega),$$ (189)

$$b_{1,1} = -P_s(k_x, \omega),$$ (190)

$$b_{2,1} = 0,$$ (191)

$$b_{3,1} = 0,$$ (192)

$$b_{4,1} = 0,$$ (193)

$$b_{5,1} = 0,$$ (194)

$$b_{6,1} = 0,$$ (195)

$$b_{7,1} = 0,$$ (196)

and

$$b_{8,1} = 0.$$ (197)

Using equations (118) - (197), the solution to the constants A_1, B_1, C_1, D_1, A_2, B_2, C_2, and D_2 can be found by

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}.$$ (198)

Additionally, the transfer function between the normal displacement at location z (when $z < b$) and the structural load can be written as

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$$T(k_x,\omega) = \frac{U_z(z,k_x,\omega)}{P_s(k_x,\omega)} = A_1(k_x,\omega)i\alpha_1\exp(i\alpha_1 z) - B_1(k_x,\omega)i\alpha_1\exp(-i\alpha_1 z) +$$

$$C_1(k_x,\omega)ik_x\exp(i\beta_1 z) + D_1(k_x,\omega)ik_x\exp(-i\beta_1 z), \qquad (199)$$

or (when $z > b$)

$$T(k_x,\omega) = \frac{U_z(z,k_x,\omega)}{P_s(k_x,\omega)} = A_2(k_x,\omega)i\alpha_2\exp(i\alpha_2 z) - B_2(k_x,\omega)i\alpha_2\exp(-i\alpha_2 z) +$$

$$C_2(k_x,\omega)ik_x\exp(i\beta_2 z) + D_2(k_x,\omega)ik_x\exp(-i\beta_2 z). \qquad (200)$$

The transfer function between the normal stress at location z (when $z < b$) and the structural load can be written as

$$T(k_x,\omega) = \frac{T_{zz}(z,k_x,\omega)}{P_s(k_x,\omega)} = A_1(k_x,\omega)\left(-\alpha_1^2\lambda_1 - 2\alpha_1^2\mu_1 - \lambda_1 k_x^2\right)\exp(i\alpha_1 z) +$$

$$B_1(k_x,\omega)\left(-\alpha_1^2\lambda_1 - 2\alpha_1^2\mu_1 - \lambda_1 k_x^2\right)\exp(-i\alpha_1 z) +$$

$$C_1(k_x,\omega)(-2k_x\beta_1\mu_1)\exp(i\beta_1 z) + D_1(k_x,\omega)(2k_x\beta_1\mu_1)\exp(-i\beta_1 z), \qquad (201)$$

or (when $z > b$)

$$T(k_x,\omega) = \frac{T_{zz}(z,k_x,\omega)}{P_s(k_x,\omega)} = A_2(k_x,\omega)\left(-\alpha_2^2\lambda_2 - 2\alpha_2^2\mu_2 - \lambda_2 k_x^2\right)\exp(i\alpha_2 z) +$$

$$B_2(k_x,\omega)\left(-\alpha_2^2\lambda_2 - 2\alpha_2^2\mu_2 - \lambda_2 k_x^2\right)\exp(-i\alpha_2 z) +$$

$$C_2(k_x,\omega)(-2k_x\beta_2\mu_2)\exp(i\beta_2 z) + D_2(k_x,\omega)(2k_x\beta_2\mu_2)\exp(-i\beta_2 z). \qquad (202)$$

Figure 11 is a plot of displacement in the z-direction (normal) divided by normal pressure versus wavenumber in the x-direction at a forcing frequency of 500 Hz evaluated at $z = c$. In this example, the sum of the thickness of the two plates is very small ($h = 0.01$ m). The solid line represents the thick plate theory using two plates (equation (200)) and the x's represent the thin plate theory (equation (68)) using $h = 0.01$ m. Additionally, the material properties of the two (thick) plates are identical so that a direct comparison between the thick and thin plate theories can be made using only a single thin plate. Parameters used to formulate this model are: bottom of plate 1 (a) is -0.010 m, intersection of plates 2 and 1 (b) is -0.005 m, top of plate 2 (c) is 0 m, Young's modulus (E) is 13.2e9 (1+0.03i) N/m^2, Poisson's ratio (v) is 0.30, and density of the plate (ρ) is 1938 kg/m^3. Only positive values of wavenumber are shown, as the function is symmetric about $k = 0$. Note that there is almost complete agreement between the thick plate theory using two similar thick plates and the thin plate theory using one plate for this specific example. Figure 12 is a plot of displacement in the z-direction (normal) divided by normal pressure versus wavenumber in the x-direction at a forcing frequency of 500 Hz evaluated at $z = c$. In this example, the sum of the thickness of the two plates is very small ($h = 0.015$ m). The solid line represents the thick plate theory using two plates (equation (200)) and the x's represent

the thin plate theory (equation (68)) using $h = 0.005$ m. Additionally, the material properties of the two (thick) plates are extremely dissimilar so that a direct comparison between the thick and thin plate theories can be made using only a single thin plate with the thickness of the stiff thick plate ($h = 0.005$ m). Parameters used to formulate this model are: bottom of plate 1 (a) is -0.015 m, intersection of plates 2 and 1 (b) is -0.005 m, top of plate 2 (c) is 0 m, Young's modulus of plate 1(E_1) is 13.2e9 (1+0.03i) N/m^2, Young's modulus of plate 2 (E_2) is 13.2e4 N/m^2, Poisson's ratio of plates 1 and 2 (v) is 0.30, and density of plates 1 and 2 (ρ) is 1938 kg/m^3. Only positive values of wavenumber are shown, as the function is symmetric about $k = 0$. Note that there is almost complete agreement between the thick plate theory using two dissimilar thick plates and the thin plate theory using one plate (matched to the stiff plate parameters) for this specific example. Figure 13 is a plot of the wavenumber-frequency (kω) surface of the system using the thick plate equations of motion (equation (200)) evaluated at $z = c$. The first plate is a stiff material and the parameters of this plate are: bottom of the plate (a) is -0.06 m, top of the plate (b) is -0.05 m, Young's modulus of the plate (E_1) is 4.55e10 (1+0.03i) N/m^2, Poisson's ratio of the plate (v_1) is 0.30, and density of the plate (ρ_1) is 7700 kg/m^3. The second plate is a soft material and the parameters of this plate are: bottom of the plate (b) is -0.05 m, top of the plate (c) is 0 m, Young's modulus of the plate (E_2) is 1e9(1+0.15i) N/m^2, Poisson's ratio of the plate (v_2) is 0.45, and density of the plate (ρ_2) is 1200 kg/m^3. The color scale to the right of the plot is in decibels. All values greater than -190 dB are displayed as -190 dB and all values less than -220 dB are displayed as -220 dB.

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Figure 11. Transfer Function of a Double Plate with No Fluid Load (Equation (200)) Using Similar Material Properties at 500 Hz
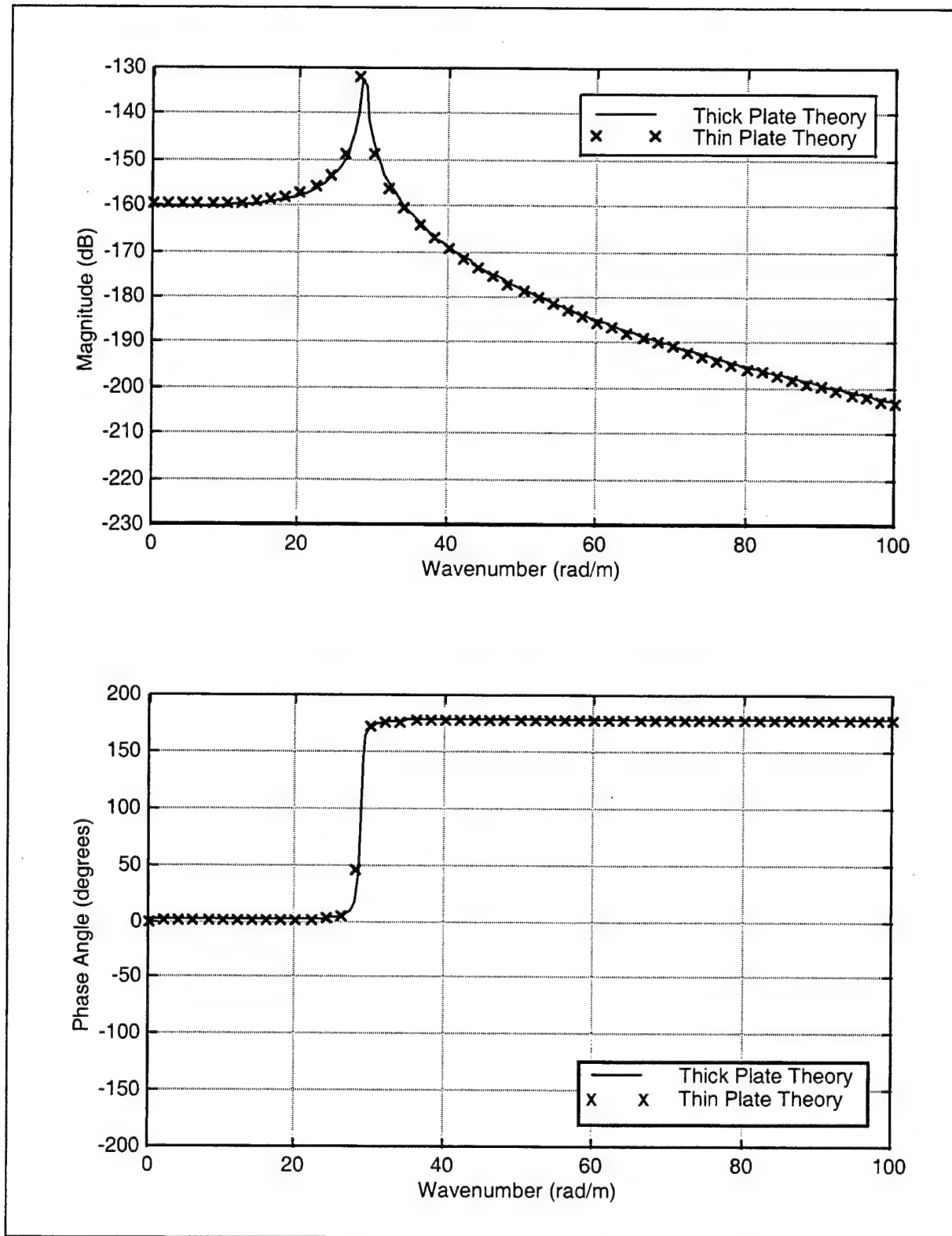
*Figure 12. Transfer Function of a Double Plate with No Fluid Load
(Equation (200)) Using Dissimilar Material Properties at 500 Hz*
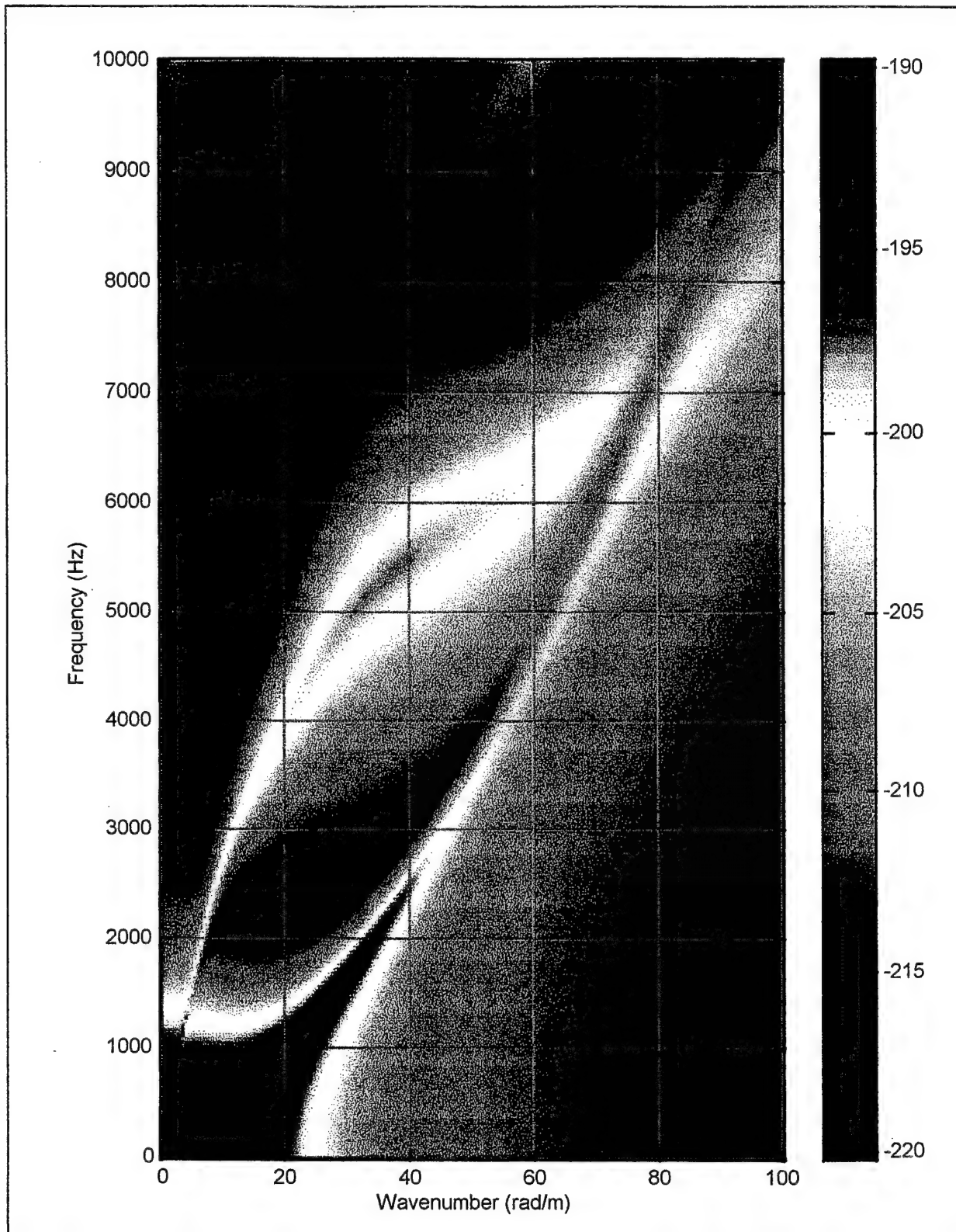
Figure 13. Transfer Surface of a Double Plate with No Fluid Load

6. DOUBLE PLATE WITH A FLUID LOAD

The fourth transfer function derived is a double plate with a fluid load. The applied load is modeled as an incident pressure wave in the fluid at definite wavenumber and frequency at location $z = c$, as shown in figure 14. The normal and tangential stresses in the system at the boundary $z = c$ are

$$\tau_{zz}(x,c,t) = (\lambda_2 + 2\mu_2)\frac{\partial u_z(x,c,t)}{\partial z} + \lambda_2 \frac{\partial u_x(x,c,t)}{\partial x} = -p_a(x,c,t), \tag{203}$$

and

$$\tau_{zx}(x,c,t) = \mu_2\left[\frac{\partial u_x(x,c,t)}{\partial z} + \frac{\partial u_z(x,c,t)}{\partial x}\right] = 0, \tag{204}$$

where the subscript 2 denotes plate 2. The interface between plates 2 and 1 requires four equations. The first two are displacement constraints, which are

$$u_z(x,b,t)\big|_{\text{plate 2}} = u_z(x,b,t)\big|_{\text{plate1}}, \tag{205}$$

and

$$u_x(x,b,t)\big|_{\text{plate 2}} = u_x(x,b,t)\big|_{\text{plate1}}. \tag{206}$$

The second two are stress constraints, which are

$$\tau_{zz}(x,b,t)\big|_{\text{plate 2}} = \tau_{zz}(x,b,t)\big|_{\text{plate1}}, \tag{207}$$

and

$$\tau_{zx}(x,b,t)\big|_{\text{plate 2}} = \tau_{zx}(x,b,t)\big|_{\text{plate1}}. \tag{208}$$

Finally, the normal and tangential stresses in the system at the boundary $z = a$ are

$$\tau_{zz}(x,a,t) = (\lambda_1 + 2\mu_1)\frac{\partial u_z(x,a,t)}{\partial z} + \lambda_1 \frac{\partial u_x(x,a,t)}{\partial x} = 0, \tag{209}$$

and

$$\tau_{zx}(x,a,t) = \mu_1\left[\frac{\partial u_x(x,a,t)}{\partial z} + \frac{\partial u_z(x,a,t)}{\partial x}\right] = 0, \tag{210}$$

where the subscript 1 denotes plate 1.
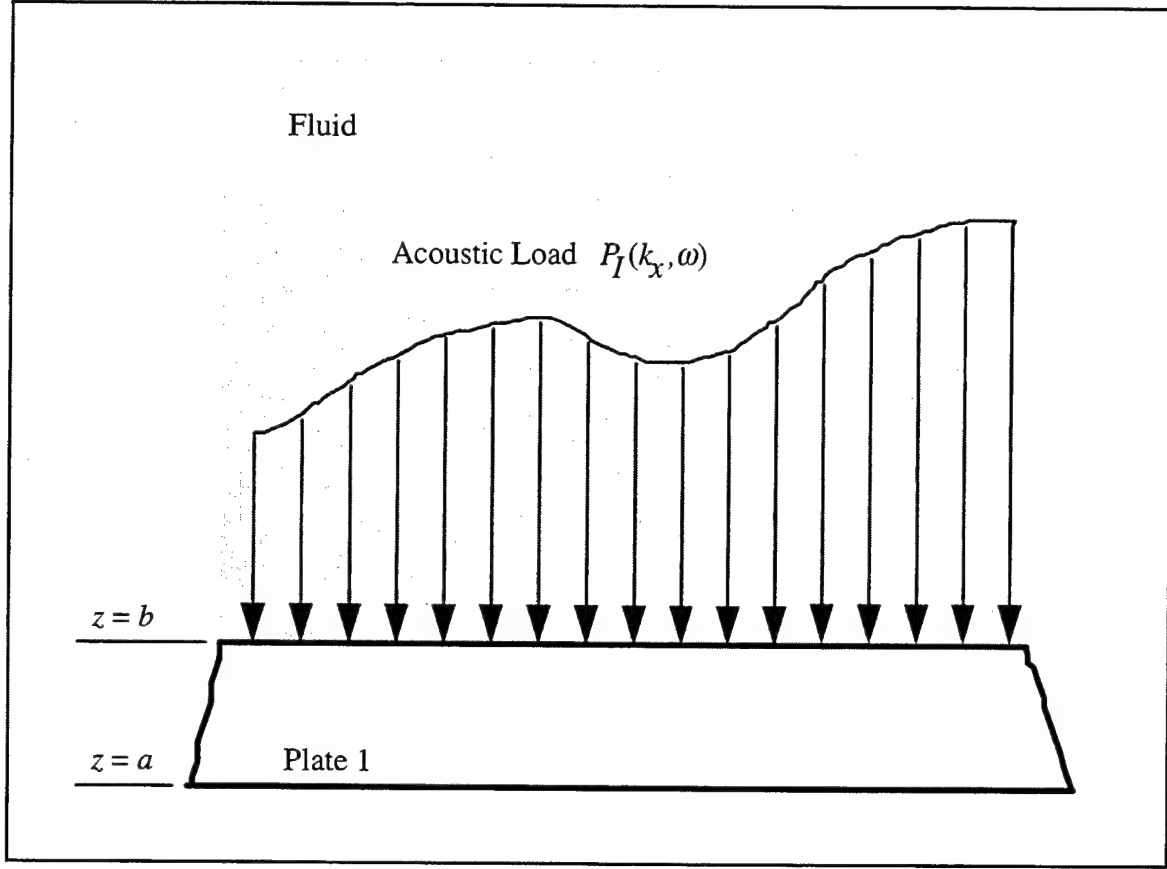
Figure 14. Double Plate with Fluid Load

Combining equations (75) - (78) and (203) - (210) yield the eight-by-eight linear system of equations

$$\mathbf{Ax} = \mathbf{b},$$

(211)

where the entries of equation (211) are

$$A_{1,1} = \left(-\alpha_2^2 \lambda_2 - 2\alpha_2^2 \mu_2 - \lambda_2 k_x^2 + \frac{\rho_f \omega^2 \alpha_2}{\gamma} \right) \exp(i\alpha_2 c),$$

(212)

$$A_{1,2} = \left(-\alpha_2^2 \lambda_2 - 2\alpha_2^2 \mu_2 - \lambda_2 k_x^2 - \frac{\rho_f \omega^2 \alpha_2}{\gamma} \right) \exp(-i\alpha_2 c),$$

(213)

$$A_{1,3} = \left(-2k_x \beta_2 \mu_2 + \frac{\rho_f \omega^2 k_x}{\gamma} \right) \exp(i\beta_2 c),$$

(214)

$$A_{1,4} = \left(2k_x\beta_2\mu_2 + \frac{\rho_f\omega^2 k_x}{\gamma} \right)\exp(-i\beta_2 c), \tag{215}$$

$$A_{1,5} = 0, \tag{216}$$

$$A_{1,6} = 0, \tag{217}$$

$$A_{1,7} = 0, \tag{218}$$

$$A_{1,8} = 0, \tag{219}$$

$$A_{2,1} = \left(-2\mu_2 k_x\alpha_2 \right)\exp(i\alpha_2 c), \tag{220}$$

$$A_{2,2} = \left(2\mu_2 k_x\alpha_2 \right)\exp(-i\alpha_2 c), \tag{221}$$

$$A_{2,3} = \left(\mu_2\beta_2^2 - \mu_2 k_x^2 \right)\exp(i\beta_2 c), \tag{222}$$

$$A_{2,4} = \left(\mu_2\beta_2^2 - \mu_2 k_x^2 \right)\exp(-i\beta_2 c), \tag{223}$$

$$A_{2,5} = 0, \tag{224}$$

$$A_{2,6} = 0, \tag{225}$$

$$A_{2,7} = 0, \tag{226}$$

$$A_{2,8} = 0, \tag{227}$$

$$A_{3,1} = \left(i\alpha_2 \right)\exp(i\alpha_2 b), \tag{228}$$

$$A_{3,2} = \left(-i\alpha_2 \right)\exp(-i\alpha_2 b), \tag{229}$$

$$A_{3,3} = \left(ik_x \right)\exp(i\beta_2 b), \tag{230}$$

$$A_{3,4} = \left(ik_x \right)\exp(-i\beta_2 b), \tag{231}$$

$$A_{3,5} = \left(-i\alpha_1 \right)\exp(i\alpha_1 b), \tag{232}$$

$$A_{3,6} = \left(i\alpha_1 \right)\exp(-i\alpha_1 b), \tag{233}$$

$$A_{3,7} = (-ik_x)\exp(i\beta_1 b),$$

(234)

$$A_{3,8} = (-ik_x)\exp(-i\beta_1 b),$$

(235)

$$A_{4,1} = (ik_x)\exp(i\alpha_2 b),$$

(236)

$$A_{4,2} = (ik_x)\exp(-i\alpha_2 b),$$

(237)

$$A_{4,3} = (-i\beta_2)\exp(i\beta_2 b),$$

(238)

$$A_{4,4} = (i\beta_2)\exp(-i\beta_2 b),$$

(239)

$$A_{4,5} = (-ik_x)\exp(i\alpha_1 b),$$

(240)

$$A_{4,6} = (-ik_x)\exp(-i\alpha_1 b),$$

(241)

$$A_{4,7} = (i\beta_1)\exp(i\beta_1 b),$$

(242)

$$A_{4,8} = (-i\beta_1)\exp(-i\beta_1 b),$$

(243)

$$A_{5,1} = \left(-\alpha_2^2\lambda_2 - 2\alpha_2^2\mu_2 - \lambda_2 k_x^2\right)\exp(i\alpha_2 b),$$

(244)

$$A_{5,2} = \left(-\alpha_2^2\lambda_2 - 2\alpha_2^2\mu_2 - \lambda_2 k_x^2\right)\exp(-i\alpha_2 b),$$

(245)

$$A_{5,3} = \left(-2k_x\beta_2\mu_2\right)\exp(i\beta_2 b),$$

(246)

$$A_{5,4} = \left(2k_x\beta_2\mu_2\right)\exp(-i\beta_2 b),$$

(247)

$$A_{5,5} = \left(\alpha_1^2\lambda_1 + 2\alpha_1^2\mu_1 + \lambda_1 k_x^2\right)\exp(i\alpha_1 b),$$

(248)

$$A_{5,6} = \left(\alpha_1^2\lambda_1 + 2\alpha_1^2\mu_1 + \lambda_1 k_x^2\right)\exp(-i\alpha_1 b),$$

(249)

$$A_{5,7} = \left(2k_x\beta_1\mu_1\right)\exp(i\beta_1 b),$$

(250)

$$A_{5,8} = \left(-2k_x\beta_1\mu_1\right)\exp(-i\beta_1 b),$$

(251)

$$A_{6,1} = \left(-2\mu_2 k_x\alpha_2\right)\exp(i\alpha_2 b),$$

(252)

$$A_{6,2} = \left(2\mu_2 k_x \alpha_2\right)\exp(-i\alpha_2 b), \tag{253}$$

$$A_{6,3} = \left(\mu_2 \beta_2^2 - \mu_2 k_x^2\right)\exp(i\beta_2 b), \tag{254}$$

$$A_{6,4} = \left(\mu_2 \beta_2^2 - \mu_2 k_x^2\right)\exp(-i\beta_2 b), \tag{255}$$

$$A_{6,5} = \left(2\mu_1 k_x \alpha_1\right)\exp(i\alpha_1 b), \tag{256}$$

$$A_{6,6} = \left(-2\mu_1 k_x \alpha_1\right)\exp(-i\alpha_1 b), \tag{257}$$

$$A_{6,7} = \left(-\mu_1 \beta_1^2 + \mu_1 k_x^2\right)\exp(i\beta_1 b), \tag{258}$$

$$A_{6,8} = \left(-\mu_1 \beta_1^2 + \mu_1 k_x^2\right)\exp(-i\beta_1 b), \tag{259}$$

$$A_{7,1} = 0, \tag{260}$$

$$A_{7,2} = 0, \tag{261}$$

$$A_{7,3} = 0, \tag{262}$$

$$A_{7,4} = 0, \tag{263}$$

$$A_{7,5} = \left(-\alpha_1^2 \lambda_1 - 2\alpha_1^2 \mu_1 - \lambda_1 k_x^2\right)\exp(i\alpha_1 a), \tag{264}$$

$$A_{7,6} = \left(-\alpha_1^2 \lambda_1 - 2\alpha_1^2 \mu_1 - \lambda_1 k_x^2\right)\exp(-i\alpha_1 a), \tag{265}$$

$$A_{7,7} = \left(-2 k_x \beta_1 \mu_1\right)\exp(i\beta_1 a), \tag{266}$$

$$A_{7,8} = \left(2 k_x \beta_1 \mu_1\right)\exp(-i\beta_1 a), \tag{267}$$

$$A_{8,1} = 0, \tag{268}$$

$$A_{8,2} = 0, \tag{269}$$

$$A_{8,3} = 0, \tag{270}$$

$$A_{8,4} = 0, \tag{271}$$

$$A_{8,5} = \left(-2\mu_1 k_x \alpha_1\right)\exp(i\alpha_1 a), \tag{272}$$

$$A_{8,6} = \left(2\mu_1 k_x \alpha_1\right)\exp(-i\alpha_1 a),\tag{273}$$

$$A_{8,7} = \left(\mu_1 \beta_1^2 - \mu_1 k_x^2\right)\exp(i\beta_1 a),\tag{274}$$

$$A_{8,8} = \left(\mu_1 \beta_1^2 - \mu_1 k_x^2\right)\exp(-i\beta_1 a),\tag{275}$$

$$x_{1,1} = A_2(k_x,\omega),\tag{276}$$

$$x_{2,1} = B_2(k_x,\omega),\tag{277}$$

$$x_{3,1} = C_2(k_x,\omega),\tag{278}$$

$$x_{4,1} = D_2(k_x,\omega),\tag{279}$$

$$x_{5,1} = A_1(k_x,\omega),\tag{280}$$

$$x_{6,1} = B_1(k_x,\omega),\tag{281}$$

$$x_{7,1} = C_1(k_x,\omega),\tag{282}$$

$$x_{8,1} = D_1(k_x,\omega),\tag{283}$$

$$b_{1,1} = -2P_I(k_x,\omega)\exp(-i\gamma c),\tag{284}$$

$$b_{2,1} = 0,\tag{285}$$

$$b_{3,1} = 0,\tag{286}$$

$$b_{4,1} = 0,\tag{287}$$

$$b_{5,1} = 0,\tag{288}$$

$$b_{6,1} = 0,\tag{289}$$

$$b_{7,1} = 0,\tag{290}$$

and

$$b_{8,1} = 0.\tag{291}$$

Using equations (212) - (291), the solution to the constants A_1, B_1, C_1, D_1, A_2, B_2, C_2, and D_2 can be found by

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}. \tag{292}$$

Additionally, the transfer function between the normal displacement at location z (when $z < b$) and the structural load can be written as

$$T(k_x, \omega) = \frac{U_z(z, k_x, \omega)}{P_I(k_x, \omega)\exp(-i\gamma c)} = A_1(k_x, \omega)i\alpha_1 \exp(i\alpha_1 z) -$$

$$B_1(k_x, \omega)i\alpha_1 \exp(-i\alpha_1 z) +$$

$$C_1(k_x, \omega)ik_x \exp(i\beta_1 z) + D_1(k_x, \omega)ik_x \exp(-i\beta_1 z), \tag{293}$$

or (when $z > b$)

$$T(k_x, \omega) = \frac{U_z(z, k_x, \omega)}{P_I(k_x, \omega)\exp(-i\gamma c)} = A_2(k_x, \omega)i\alpha_2 \exp(i\alpha_2 z) -$$

$$B_2(k_x, \omega)i\alpha_2 \exp(-i\alpha_2 z) +$$

$$C_2(k_x, \omega)ik_x \exp(i\beta_2 z) + D_2(k_x, \omega)ik_x \exp(-i\beta_2 z). \tag{294}$$

The transfer function between the normal stress at location z (when $z < b$) and the structural load can be written as

$$T(k_x, \omega) = \frac{U_z(z, k_x, \omega)}{P_I(k_x, \omega)\exp(-i\gamma c)} = A_2(k_x, \omega)i\alpha_2 \exp(i\alpha_2 z) -$$

$$B_2(k_x, \omega)i\alpha_2 \exp(-i\alpha_2 z) +$$

$$C_2(k_x, \omega)ik_x \exp(i\beta_2 z) + D_2(k_x, \omega)ik_x \exp(-i\beta_2 z). \tag{295}$$

or (when $z > b$)

$$T(k_x, \omega) = \frac{T_{zz}(z, k_x, \omega)}{P_I(k_x, \omega)\exp(-i\gamma c)} = A_2(k_x, \omega)\left(-\alpha_2^2\lambda_2 - 2\alpha_2^2\mu_2 - \lambda_2 k_x^2\right)\exp(i\alpha_2 z) +$$

$$B_2(k_x, \omega)\left(-\alpha_2^2\lambda_2 - 2\alpha_2^2\mu_2 - \lambda_2 k_x^2\right)\exp(-i\alpha_2 z) +$$

$$C_2(k_x, \omega)(-2k_x\beta_2\mu_2)\exp(i\beta_2 z) + D_2(k_x, \omega)(2k_x\beta_2\mu_2)\exp(-i\beta_2 z). \tag{296}$$

Figure 15 is a plot of the displacement in the z-direction (normal) divided by normal pressure versus wavenumber in the x-direction at a forcing frequency of 500 Hz evaluated at $z = c$. In this example, the sum of the thickness of the two plates is very small ($h = 0.01$ m). The solid line represents the thick plate theory using two plates (equation (294)) and the x's represent the thin plate theory (equation (107)) using $h = 0.01$ m. Additionally, the material properties of the two (thick) plates are identical so that a direct comparison between the thick and thin plate theories can be made using only a single thin plate. Parameters used to formulate this model are: bottom of plate 1 (a) is -0.010 m, intersection of plates 2 and 1 (b) is -0.005 m, top of plate 2 (c) is 0 m, Young's modulus (E) is 13.2e9 (1+0.03i) N/m^2, Poisson's ratio (v) is 0.30, and density of the plate (ρ) is 1938 kg/m^3. Only positive values of wavenumber are shown, as the function is symmetric about $k = 0$. Note that there is almost complete agreement between the thick plate theory using two similar thick plates and the thin plate theory using one plate for this specific example. Figure 16 is a plot of the displacement in the z-direction (normal) divided by normal pressure versus wavenumber in the x-direction at a forcing frequency of 500 Hz evaluated at $z = c$. In this example, the sum of the thickness of the two plates is very small ($h = 0.015$ m). The solid line represents the thick plate theory using two plates (equation (294)) and the x's represent the thin plate theory (equation (107)) using $h = 0.005$ m. Additionally, the material properties of the two (thick) plates are extremely dissimilar so that a direct comparison between the thick and thin plate theories can be made using only a single thin plate with the thickness of the stiff thick plate ($h = 0.005$ m). Parameters used to formulate this model are: bottom of plate 1 (a) is -0.015 m, intersection of plates 2 and 1 (b) is -0.005 m, top of plate 2 (c) is 0 m, Young's modulus of plate 2 (E_2) is 13.2e9 (1+0.03i) N/m^2, Young's modulus of plate 1 (E_1) is 13.2e4 N/m^2, Poisson's ratio of plates 1 and 2 (v) is 0.30, and density of plates 1 and 2 (ρ) is 1938 kg/m^3. Only positive values of wavenumber are shown, as the function is symmetric about $k = 0$. Note that there is almost complete agreement between the thick plate theory using two dissimilar thick plates and the thin plate theory using one plate (matched to the stiff plate parameters) for this specific example. Figure 17 is a plot of the wavenumber-frequency ($k\omega$) surface of the system using the thick plate equations of motion (equation (294)) evaluated at $z = c$. The first plate is a stiff material and the parameters of this plate are: bottom of the plate (a) is -0.06 m, top of the plate (b) is -0.05 m, Young's modulus of the plate (E_1) is 4.55e10 (1+0.03i) N/m^2, Poisson's ratio of the plate (v_1) is 0.30, and density of the plate (ρ_1) is 7700 kg/m^3. The second plate is a soft material and the parameters of this plate are: bottom of the plate (b) is -0.05 m, top of the plate (c) is 0 m, Young's modulus of the plate (E_2) is 1e9 (1+0.15i) N/m^2, Poisson's ratio of the plate (v_2) is 0.45, and density of the plate (ρ_2) is 1200 kg/m^3. The color scale to the right of the plot is in decibels. All values greater than -190 dB are displayed as -190 dB and all values less than -220 dB are displayed as -220 dB.
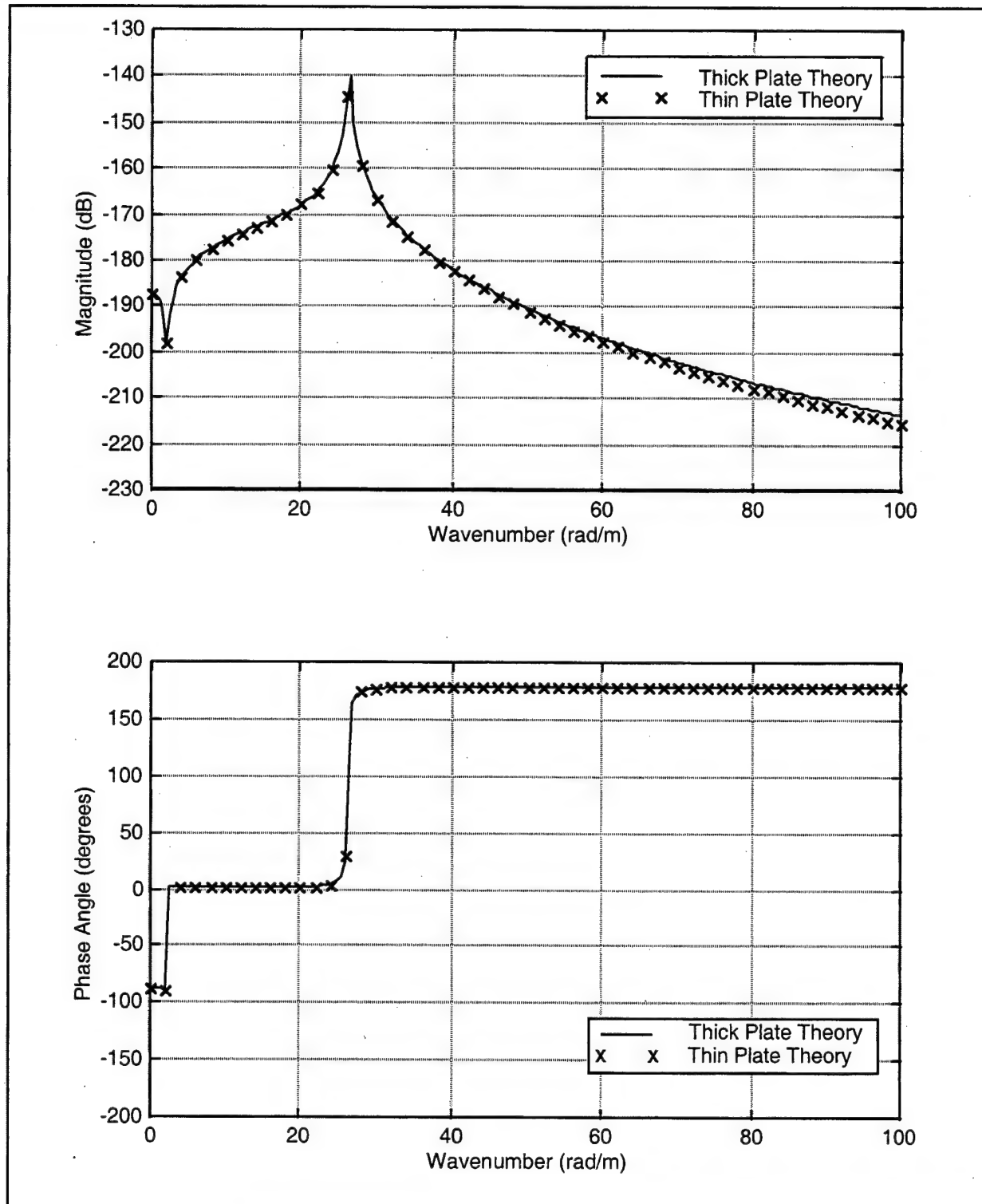
Figure 15. Transfer Function of a Double Plate with Fluid Load (Equation (294)) Using Similar Material Properties at 500 Hz
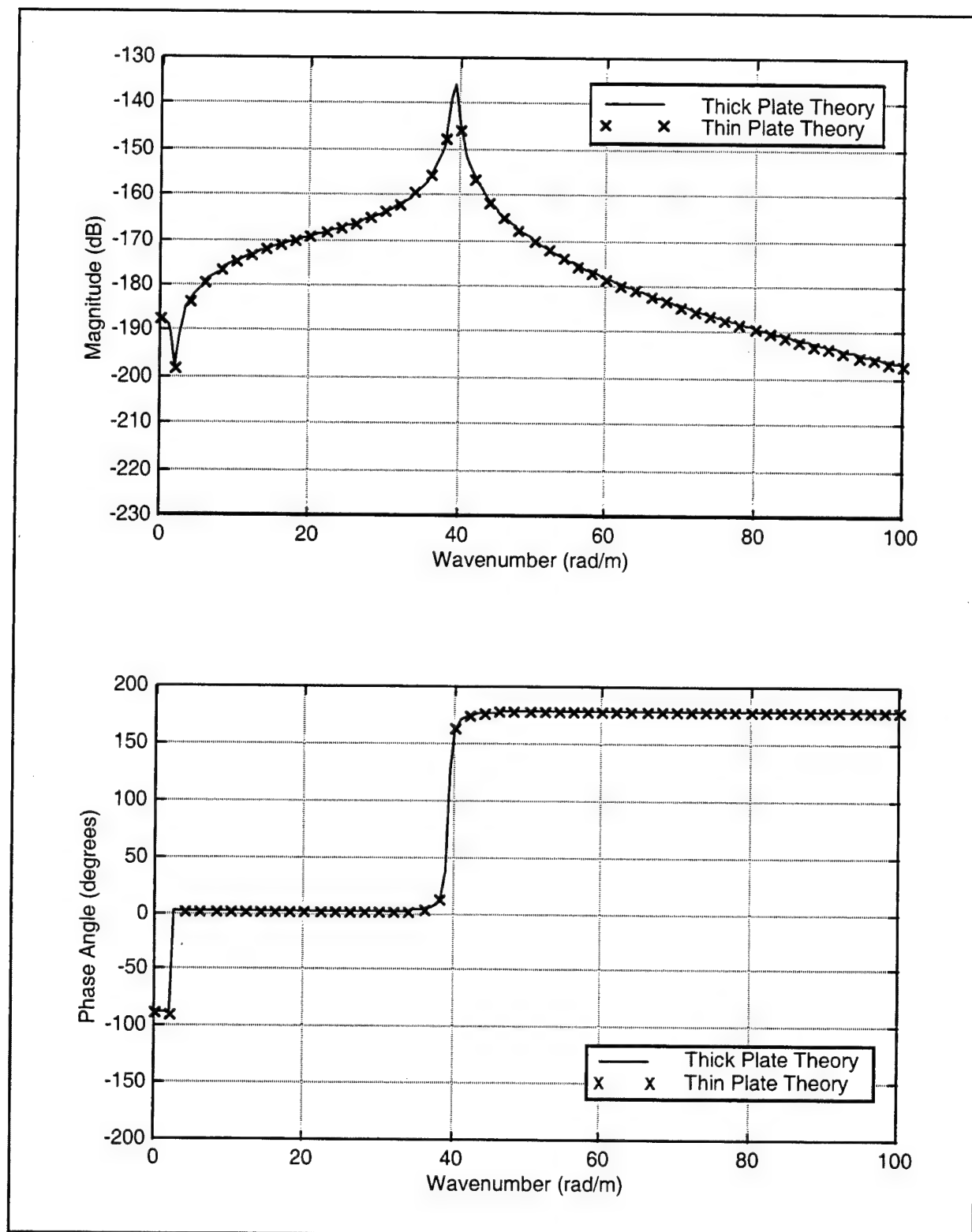
Figure 16. Transfer Function of a Double Plate with Fluid Load (Equation (294)) Using Dissimilar Material Properties at 500 Hz
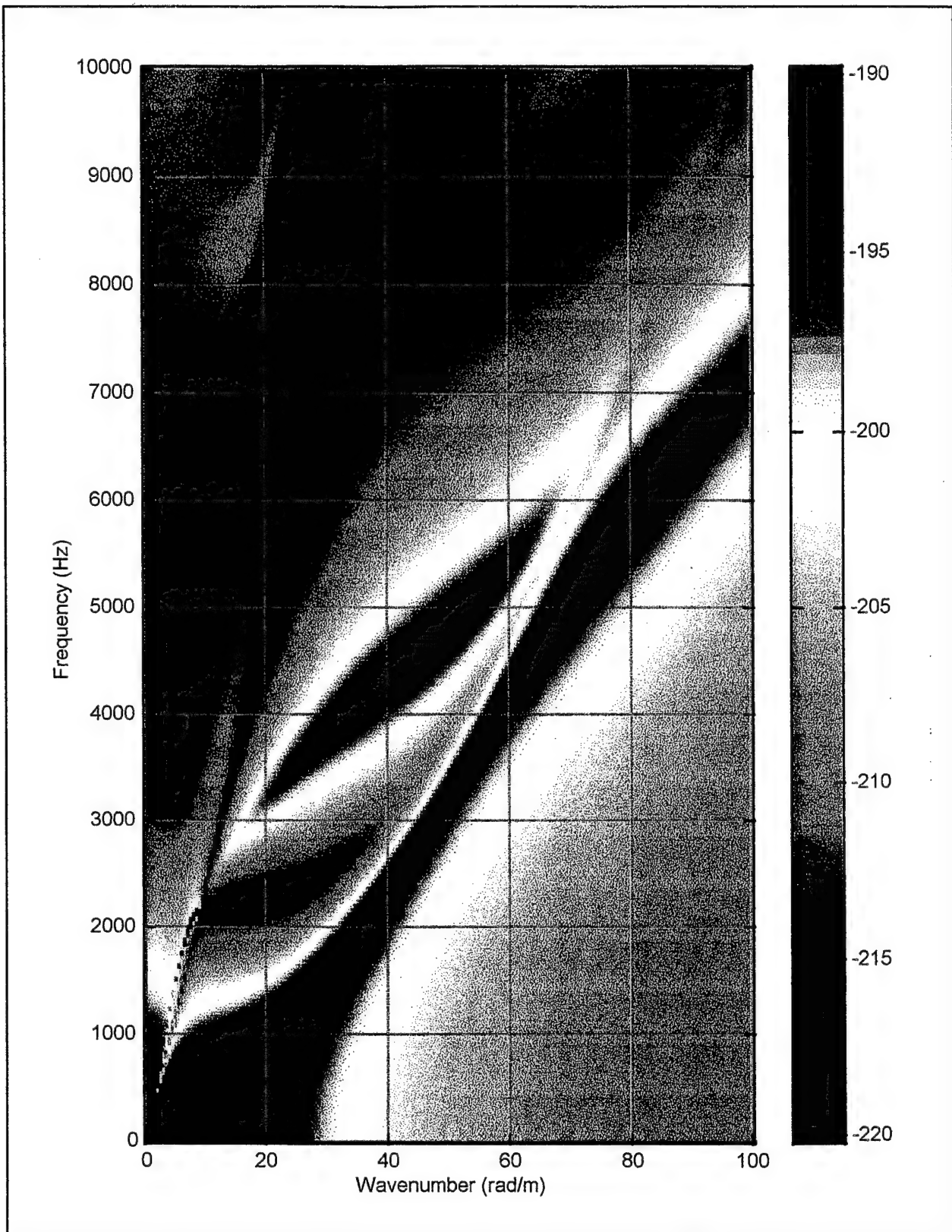
Figure 17. Transfer Surface of a Double Plate with Fluid Load

7. TRIPLE PLATE WITH NO FLUID LOAD

The fifth transfer function derived is a triple plate with no fluid load. The structural load consists of an applied pressure at definite wavenumber and frequency at location $z = d$, as shown in figure 18. The normal and tangential stresses in the system at the boundary $z = d$ are

$$\tau_{zz}(x,d,t) = (\lambda_3 + 2\mu_3)\frac{\partial u_z(x,d,t)}{\partial z} + \lambda_3 \frac{\partial u_x(x,d,t)}{\partial x} = -p_s(x,d,t), \tag{297}$$

and

$$\tau_{zx}(x,d,t) = \mu_3 \left[\frac{\partial u_x(x,d,t)}{\partial z} + \frac{\partial u_z(x,d,t)}{\partial x} \right] = 0, \tag{298}$$

where the subscript 3 denotes plate 3. The interface between plates 3 and 2 requires four equations. The first two are displacement constraints, which are

$$u_z(x,c,t)\big|_{\text{plate 3}} = u_z(x,c,t)\big|_{\text{plate 2}}, \tag{299}$$

and

$$u_x(x,c,t)\big|_{\text{plate 3}} = u_x(x,c,t)\big|_{\text{plate 2}}. \tag{300}$$

The second two are stress constraints, which are

$$\tau_{zz}(x,c,t)\big|_{\text{plate 3}} = \tau_{zz}(x,c,t)\big|_{\text{plate 2}}, \tag{301}$$

and

$$\tau_{zx}(x,c,t)\big|_{\text{plate 3}} = \tau_{zx}(x,c,t)\big|_{\text{plate 2}}. \tag{302}$$

The interface between plates 2 and 1 requires four equations. The first two are displacement constraints, which are

$$u_z(x,b,t)\big|_{\text{plate 2}} = u_z(x,b,t)\big|_{\text{plate 1}}, \tag{303}$$

and

$$u_x(x,b,t)\big|_{\text{plate 2}} = u_x(x,b,t)\big|_{\text{plate 1}}. \tag{304}$$

The second two are stress constraints, which are

$$\tau_{zz}(x,b,t)\big|_{\text{plate 2}} = \tau_{zz}(x,b,t)\big|_{\text{plate 1}}, \tag{305}$$

and
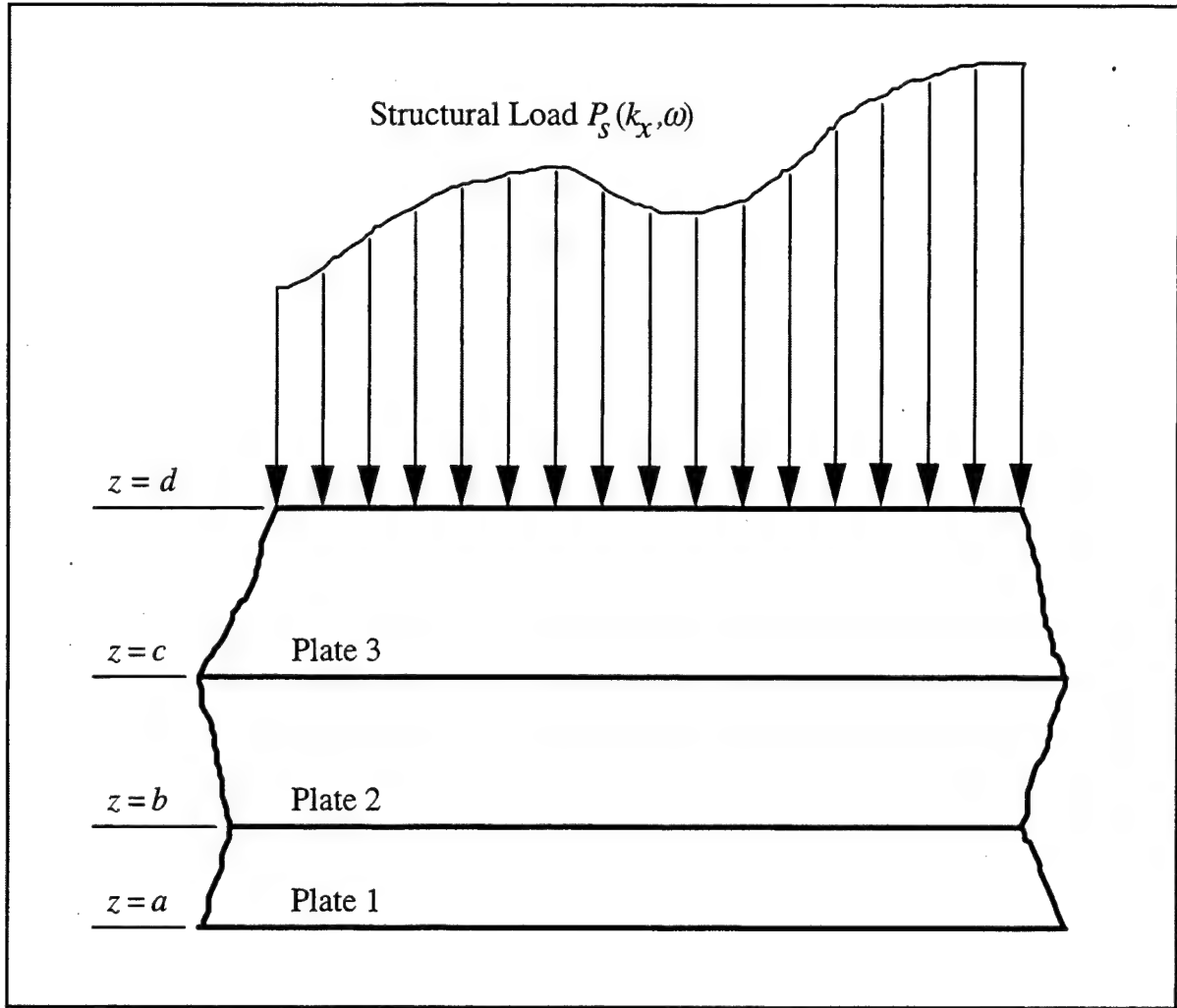
Figure 18. Triple Plate with No Fluid Load

$$\tau_{zx}(x,b,t)\big|_{\text{plate 2}} = \tau_{zx}(x,b,t)\big|_{\text{plate 1}} \, . \tag{306}$$

Finally, the normal and tangential stresses in the system at the boundary $z = a$ are

$$\tau_{zz}(x,a,t) = (\lambda_1 + 2\mu_1)\frac{\partial u_z(x,a,t)}{\partial z} + \lambda_1 \frac{\partial u_x(x,a,t)}{\partial x} = 0 \, , \tag{307}$$

and

$$\tau_{zx}(x,a,t) = \mu_1\left[\frac{\partial u_x(x,a,t)}{\partial z} + \frac{\partial u_z(x,a,t)}{\partial x}\right] = 0 \, , \tag{308}$$

where the subscript 1 denotes plate 1.

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Combining equations (297) - (308) yields the twelve-by-twelve linear system of equations

$$\mathbf{Ax} = \mathbf{b},\tag{309}$$

where the entries of equation (309) are

$$A_{1,1} = \left(-\alpha_3^2\lambda_3 - 2\alpha_3^2\mu_3 - \lambda_3 k_x^2\right)\exp(i\alpha_3 d),\tag{310}$$

$$A_{1,2} = \left(-\alpha_3^2\lambda_3 - 2\alpha_3^2\mu_3 - \lambda_3 k_x^2\right)\exp(-i\alpha_3 d),\tag{311}$$

$$A_{1,3} = \left(-2k_x\beta_3\mu_3\right)\exp(i\beta_3 d),\tag{312}$$

$$A_{1,4} = \left(2k_x\beta_3\mu_3\right)\exp(-i\beta_3 d),\tag{313}$$

$$A_{1,5} = 0,\tag{314}$$

$$A_{1,6} = 0,\tag{315}$$

$$A_{1,7} = 0,\tag{316}$$

$$A_{1,8} = 0,\tag{317}$$

$$A_{1,9} = 0,\tag{318}$$

$$A_{1,10} = 0,\tag{319}$$

$$A_{1,11} = 0,\tag{320}$$

$$A_{1,12} = 0,\tag{321}$$

$$A_{2,1} = \left(-2\mu_3 k_x\alpha_3\right)\exp(i\alpha_3 d),\tag{322}$$

$$A_{2,2} = \left(2\mu_3 k_x\alpha_3\right)\exp(-i\alpha_3 d),\tag{323}$$

$$A_{2,3} = \left(\mu_3\beta_3^2 - \mu_3 k_x^2\right)\exp(i\beta_3 d),\tag{324}$$

$$A_{2,4} = \left(\mu_3\beta_3^2 - \mu_3 k_x^2\right)\exp(-i\beta_3 d),\tag{325}$$

$$A_{2,5} = 0,\tag{326}$$

$$A_{2,6} = 0, \tag{327}$$

$$A_{2,7} = 0, \tag{328}$$

$$A_{2,8} = 0, \tag{329}$$

$$A_{2,9} = 0, \tag{330}$$

$$A_{2,10} = 0, \tag{331}$$

$$A_{2,11} = 0, \tag{332}$$

$$A_{2,12} = 0, \tag{333}$$

$$A_{3,1} = (i\alpha_3)\exp(i\alpha_3 c), \tag{334}$$

$$A_{3,2} = (-i\alpha_3)\exp(-i\alpha_3 c), \tag{335}$$

$$A_{3,3} = (ik_x)\exp(i\beta_3 c), \tag{336}$$

$$A_{3,4} = (ik_x)\exp(-i\beta_3 c), \tag{337}$$

$$A_{3,5} = (-i\alpha_2)\exp(i\alpha_2 c), \tag{338}$$

$$A_{3,6} = (i\alpha_2)\exp(-i\alpha_2 c), \tag{339}$$

$$A_{3,7} = (-ik_x)\exp(i\beta_2 c), \tag{340}$$

$$A_{3,8} = (-ik_x)\exp(-i\beta_2 c), \tag{341}$$

$$A_{3,9} = 0, \tag{342}$$

$$A_{3,10} = 0, \tag{343}$$

$$A_{3,11} = 0, \tag{344}$$

$$A_{3,12} = 0, \tag{345}$$

$$A_{4,1} = (ik_x)\exp(i\alpha_3 c), \tag{346}$$

$$A_{4,2} = \left(ik_x\right)\exp(-i\alpha_3 c),\qquad(347)$$

$$A_{4,3} = \left(-i\beta_3\right)\exp(i\beta_3 c),\qquad(348)$$

$$A_{4,4} = \left(i\beta_3\right)\exp(-i\beta_3 c),\qquad(349)$$

$$A_{4,5} = \left(-ik_x\right)\exp(i\alpha_2 c),\qquad(350)$$

$$A_{4,6} = \left(-ik_x\right)\exp(-i\alpha_2 c),\qquad(351)$$

$$A_{4,7} = \left(i\beta_2\right)\exp(i\beta_2 c),\qquad(352)$$

$$A_{4,8} = \left(-i\beta_2\right)\exp(-i\beta_2 c),\qquad(353)$$

$$A_{4,9} = 0,\qquad(354)$$

$$A_{4,10} = 0,\qquad(355)$$

$$A_{4,11} = 0,\qquad(356)$$

$$A_{4,12} = 0,\qquad(357)$$

$$A_{5,1} = \left(-\alpha_3^2\lambda_3 - 2\alpha_3^2\mu_3 - \lambda_3 k_x^2\right)\exp(i\alpha_3 c),\qquad(358)$$

$$A_{5,2} = \left(-\alpha_3^2\lambda_3 - 2\alpha_3^2\mu_3 - \lambda_3 k_x^2\right)\exp(-i\alpha_3 c),\qquad(359)$$

$$A_{5,3} = \left(-2k_x\beta_3\mu_3\right)\exp(i\beta_3 c),\qquad(360)$$

$$A_{5,4} = \left(2k_x\beta_3\mu_3\right)\exp(-i\beta_3 c),\qquad(361)$$

$$A_{5,5} = \left(\alpha_2^2\lambda_2 + 2\alpha_2^2\mu_2 + \lambda_2 k_x^2\right)\exp(i\alpha_2 c),\qquad(362)$$

$$A_{5,6} = \left(\alpha_2^2\lambda_2 + 2\alpha_2^2\mu_2 + \lambda_2 k_x^2\right)\exp(-i\alpha_2 c),\qquad(363)$$

$$A_{5,7} = \left(2k_x\beta_2\mu_2\right)\exp(i\beta_2 c),\qquad(364)$$

$$A_{5,8} = \left(-2k_x\beta_2\mu_2\right)\exp(-i\beta_2 c),\qquad(365)$$

$$A_{5,9} = 0,\qquad(366)$$

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$$A_{5,10} = 0,$$ (367)

$$A_{5,11} = 0,$$ (368)

$$A_{5,12} = 0,$$ (369)

$$A_{6,1} = (-2\mu_3 k_x \alpha_3) \exp(i\alpha_3 c),$$ (370)

$$A_{6,2} = (2\mu_3 k_x \alpha_3) \exp(-i\alpha_3 c),$$ (371)

$$A_{6,3} = (\mu_3 \beta_3^2 - \mu_3 k_x^2) \exp(i\beta_3 c),$$ (372)

$$A_{6,4} = (\mu_3 \beta_3^2 - \mu_3 k_x^2) \exp(-i\beta_3 c),$$ (373)

$$A_{6,5} = (2\mu_2 k_x \alpha_2) \exp(i\alpha_2 c),$$ (374)

$$A_{6,6} = (-2\mu_2 k_x \alpha_2) \exp(-i\alpha_2 c),$$ (375)

$$A_{6,7} = (-\mu_2 \beta_2^2 + \mu_2 k_x^2) \exp(i\beta_2 c),$$ (376)

$$A_{6,8} = (-\mu_2 \beta_2^2 + \mu_2 k_x^2) \exp(-i\beta_2 c),$$ (377)

$$A_{6,9} = 0,$$ (378)

$$A_{6,10} = 0,$$ (379)

$$A_{6,11} = 0,$$ (380)

$$A_{6,12} = 0,$$ (381)

$$A_{7,1} = 0,$$ (382)

$$A_{7,2} = 0,$$ (383)

$$A_{7,3} = 0,$$ (384)

$$A_{7,4} = 0,$$ (385)

$$A_{7,5} = (i\alpha_2) \exp(i\alpha_2 b),$$ (386)

$$A_{7,6} = (-\mathrm{i}\alpha_2)\exp(-\mathrm{i}\alpha_2 b), \tag{387}$$

$$A_{7,7} = (\mathrm{i}k_x)\exp(\mathrm{i}\beta_2 b), \tag{388}$$

$$A_{7,8} = (\mathrm{i}k_x)\exp(-\mathrm{i}\beta_2 b), \tag{389}$$

$$A_{7,9} = (-\mathrm{i}\alpha_1)\exp(\mathrm{i}\alpha_1 b), \tag{390}$$

$$A_{7,10} = (\mathrm{i}\alpha_1)\exp(-\mathrm{i}\alpha_1 b), \tag{391}$$

$$A_{7,11} = (-\mathrm{i}k_x)\exp(\mathrm{i}\beta_1 b), \tag{392}$$

$$A_{7,12} = (-\mathrm{i}k_x)\exp(-\mathrm{i}\beta_1 b), \tag{393}$$

$$A_{8,1} = 0, \tag{394}$$

$$A_{8,2} = 0, \tag{395}$$

$$A_{8,3} = 0, \tag{396}$$

$$A_{8,4} = 0, \tag{397}$$

$$A_{8,5} = (\mathrm{i}k_x)\exp(\mathrm{i}\alpha_2 b), \tag{398}$$

$$A_{8,6} = (\mathrm{i}k_x)\exp(-\mathrm{i}\alpha_2 b), \tag{399}$$

$$A_{8,7} = (-\mathrm{i}\beta_2)\exp(\mathrm{i}\beta_2 b), \tag{400}$$

$$A_{8,8} = (\mathrm{i}\beta_2)\exp(-\mathrm{i}\beta_2 b), \tag{401}$$

$$A_{8,9} = (-\mathrm{i}k_x)\exp(\mathrm{i}\alpha_1 b), \tag{402}$$

$$A_{8,10} = (-\mathrm{i}k_x)\exp(-\mathrm{i}\alpha_1 b), \tag{403}$$

$$A_{8,11} = (\mathrm{i}\beta_1)\exp(\mathrm{i}\beta_1 b), \tag{404}$$

$$A_{8,12} = (-\mathrm{i}\beta_1)\exp(-\mathrm{i}\beta_1 b), \tag{405}$$

$$A_{9,1} = 0, \tag{406}$$

$$A_{9,2} = 0 , \tag{407}$$

$$A_{9,3} = 0 , \tag{408}$$

$$A_{9,4} = 0 , \tag{409}$$

$$A_{9,5} = \left(-\alpha_2^2\lambda_2 - 2\alpha_2^2\mu_2 - \lambda_2 k_x^2\right)\exp(i\alpha_2 b) , \tag{410}$$

$$A_{9,6} = \left(-\alpha_2^2\lambda_2 - 2\alpha_2^2\mu_2 - \lambda_2 k_x^2\right)\exp(-i\alpha_2 b) , \tag{411}$$

$$A_{9,7} = \left(-2k_x\beta_2\mu_2\right)\exp(i\beta_2 b) , \tag{412}$$

$$A_{9,8} = \left(2k_x\beta_2\mu_2\right)\exp(-i\beta_2 b) , \tag{413}$$

$$A_{9,9} = \left(\alpha_1^2\lambda_1 + 2\alpha_1^2\mu_1 + \lambda_1 k_x^2\right)\exp(i\alpha_1 b) , \tag{414}$$

$$A_{9,10} = \left(\alpha_1^2\lambda_1 + 2\alpha_1^2\mu_1 + \lambda_1 k_x^2\right)\exp(-i\alpha_1 b) , \tag{415}$$

$$A_{9,11} = \left(2k_x\beta_1\mu_1\right)\exp(i\beta_1 b) , \tag{416}$$

$$A_{9,12} = \left(-2k_x\beta_1\mu_1\right)\exp(-i\beta_1 b) , \tag{417}$$

$$A_{10,1} = 0 , \tag{418}$$

$$A_{10,2} = 0 , \tag{419}$$

$$A_{10,3} = 0 , \tag{420}$$

$$A_{10,4} = 0 , \tag{421}$$

$$A_{10,5} = \left(-2\mu_2 k_x\alpha_2\right)\exp(i\alpha_2 b) , \tag{422}$$

$$A_{10,6} = \left(2\mu_2 k_x\alpha_2\right)\exp(-i\alpha_2 b) , \tag{423}$$

$$A_{10,7} = \left(\mu_2\beta_2^2 - \mu_2 k_x^2\right)\exp(i\beta_2 b) , \tag{424}$$

$$A_{10,8} = \left(\mu_2\beta_2^2 - \mu_2 k_x^2\right)\exp(-i\beta_2 b) , \tag{425}$$

$$A_{10,9} = \left(2\mu_1 k_x\alpha_1\right)\exp(i\alpha_1 b) , \tag{426}$$

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$$A_{10,10} = \left(-2\mu_1 k_x \alpha_1\right)\exp(-i\alpha_1 b), \tag{427}$$

$$A_{10,11} = \left(-\mu_1 \beta_1^2 + \mu_1 k_x^2\right)\exp(i\beta_1 b), \tag{428}$$

$$A_{10,12} = \left(-\mu_1 \beta_1^2 + \mu_1 k_x^2\right)\exp(-i\beta_1 b), \tag{429}$$

$$A_{11,1} = 0, \tag{430}$$

$$A_{11,2} = 0, \tag{431}$$

$$A_{11,3} = 0, \tag{432}$$

$$A_{11,4} = 0, \tag{433}$$

$$A_{11,5} = 0, \tag{434}$$

$$A_{11,6} = 0, \tag{435}$$

$$A_{11,7} = 0, \tag{436}$$

$$A_{11,8} = 0, \tag{437}$$

$$A_{11,9} = \left(-\alpha_1^2 \lambda_1 - 2\alpha_1^2 \mu_1 - \lambda_1 k_x^2\right)\exp(i\alpha_1 a), \tag{438}$$

$$A_{11,10} = \left(-\alpha_1^2 \lambda_1 - 2\alpha_1^2 \mu_1 - \lambda_1 k_x^2\right)\exp(-i\alpha_1 a), \tag{439}$$

$$A_{11,11} = \left(-2k_x \beta_1 \mu_1\right)\exp(i\beta_1 a), \tag{440}$$

$$A_{11,12} = \left(2k_x \beta_1 \mu_1\right)\exp(-i\beta_1 a), \tag{441}$$

$$A_{12,1} = 0, \tag{442}$$

$$A_{12,2} = 0, \tag{443}$$

$$A_{12,3} = 0, \tag{444}$$

$$A_{12,4} = 0, \tag{445}$$

$$A_{12,5} = 0, \tag{446}$$

$$A_{12,6} = 0, \tag{447}$$

$$A_{12,7} = 0, \tag{448}$$

$$A_{12,8} = 0, \tag{449}$$

$$A_{12,9} = \left(-2\mu_1 k_x \alpha_1\right)\exp(i\alpha_1 a), \tag{450}$$

$$A_{12,10} = \left(2\mu_1 k_x \alpha_1\right)\exp(-i\alpha_1 a), \tag{451}$$

$$A_{12,11} = \left(\mu_1 \beta_1^2 - \mu_1 k_x^2\right)\exp(i\beta_1 a), \tag{452}$$

$$A_{12,12} = \left(\mu_1 \beta_1^2 - \mu_1 k_x^2\right)\exp(-i\beta_1 a), \tag{453}$$

$$x_{1,1} = A_3(k_x, \omega), \tag{454}$$

$$x_{2,1} = B_3(k_x, \omega), \tag{455}$$

$$x_{3,1} = C_3(k_x, \omega), \tag{456}$$

$$x_{4,1} = D_3(k_x, \omega), \tag{457}$$

$$x_{5,1} = A_2(k_x, \omega), \tag{458}$$

$$x_{6,1} = B_2(k_x, \omega), \tag{459}$$

$$x_{7,1} = C_2(k_x, \omega), \tag{460}$$

$$x_{8,1} = D_2(k_x, \omega), \tag{461}$$

$$x_{9,1} = A_1(k_x, \omega), \tag{462}$$

$$x_{10,1} = B_1(k_x, \omega), \tag{463}$$

$$x_{11,1} = C_1(k_x, \omega), \tag{464}$$

$$x_{12,1} = D_1(k_x, \omega), \tag{465}$$

$$b_{1,1} = -P_s(k_x, \omega), \tag{466}$$

$$b_{2,1} = 0, \tag{467}$$

$$b_{3,1} = 0, \tag{468}$$

$$b_{4,1} = 0, \tag{469}$$

$$b_{5,1} = 0, \tag{470}$$

$$b_{6,1} = 0, \tag{471}$$

$$b_{7,1} = 0, \tag{472}$$

$$b_{8,1} = 0, \tag{473}$$

$$b_{9,1} = 0, \tag{474}$$

$$b_{10,1} = 0, \tag{475}$$

$$b_{11,1} = 0, \tag{476}$$

and

$$b_{12,1} = 0. \tag{477}$$

Using equations (310) - (477), the solution to the constants A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2, A_3, B_3, C_3, and D_3 can be found by

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}. \tag{478}$$

Additionally, the transfer function between the normal displacement at location z (when $z < b$) and the structural load can be written as

$$T(k_x, \omega) = \frac{U_z(z, k_x, \omega)}{P_s(k_x, \omega)} = A_1(k_x, \omega)i\alpha_1 \exp(i\alpha_1 z) - B_1(k_x, \omega)i\alpha_1 \exp(-i\alpha_1 z) +$$

$$C_1(k_x, \omega)ik_x \exp(i\beta_1 z) + D_1(k_x, \omega)ik_x \exp(-i\beta_1 z), \tag{479}$$

or (when $z > b$ and $z < c$)

$$T(k_x, \omega) = \frac{U_z(z, k_x, \omega)}{P_s(k_x, \omega)} = A_2(k_x, \omega)i\alpha_2 \exp(i\alpha_2 z) - B_2(k_x, \omega)i\alpha_2 \exp(-i\alpha_2 z) +$$

$$C_2(k_x, \omega)ik_x \exp(i\beta_2 z) + D_2(k_x, \omega)ik_x \exp(-i\beta_2 z), \tag{480}$$

or (when $z > c$)

$$T(k_x, \omega) = \frac{U_z(z, k_x, \omega)}{P_s(k_x, \omega)} = A_3(k_x, \omega)i\alpha_3 \exp(i\alpha_3 z) - B_3(k_x, \omega)i\alpha_3 \exp(-i\alpha_3 z) +$$

$$C_3(k_x, \omega)ik_x \exp(i\beta_3 z) + D_3(k_x, \omega)ik_x \exp(-i\beta_3 z). \tag{481}$$

The transfer function between the normal stress at location z (when $z < b$) and the structural load can be written as

$$T(k_x, \omega) = \frac{T_{zz}(z, k_x, \omega)}{P_s(k_x, \omega)} = A_1(k_x, \omega)\left(-\alpha_1^2\lambda_1 - 2\alpha_1^2\mu_1 - \lambda_1 k_x^2\right)\exp(i\alpha_1 z) +$$

$$B_1(k_x, \omega)\left(-\alpha_1^2\lambda_1 - 2\alpha_1^2\mu_1 - \lambda_1 k_x^2\right)\exp(-i\alpha_1 z) +$$

$$C_1(k_x, \omega)(-2k_x\beta_1\mu_1)\exp(i\beta_1 z) + D_1(k_x, \omega)(2k_x\beta_1\mu_1)\exp(-i\beta_1 z), \tag{482}$$

or (when $z > b$ and $z < c$)

$$T(k_x, \omega) = \frac{T_{zz}(z, k_x, \omega)}{P_s(k_x, \omega)} = A_2(k_x, \omega)\left(-\alpha_2^2\lambda_2 - 2\alpha_2^2\mu_2 - \lambda_2 k_x^2\right)\exp(i\alpha_2 z) +$$

$$B_2(k_x, \omega)\left(-\alpha_2^2\lambda_2 - 2\alpha_2^2\mu_2 - \lambda_2 k_x^2\right)\exp(-i\alpha_2 z) +$$

$$C_2(k_x, \omega)(-2k_x\beta_2\mu_2)\exp(i\beta_2 z) + D_2(k_x, \omega)(2k_x\beta_2\mu_2)\exp(-i\beta_2 z), \tag{483}$$

or (when $z > c$)

$$T(k_x, \omega) = \frac{T_{zz}(z, k_x, \omega)}{P_s(k_x, \omega)} = A_3(k_x, \omega)\left(-\alpha_3^2\lambda_3 - 2\alpha_3^2\mu_3 - \lambda_3 k_x^2\right)\exp(i\alpha_3 z) +$$

$$B_3(k_x, \omega)\left(-\alpha_3^2\lambda_3 - 2\alpha_3^2\mu_3 - \lambda_3 k_x^2\right)\exp(-i\alpha_3 z) +$$

$$C_3(k_x, \omega)(-2k_x\beta_3\mu_3)\exp(i\beta_3 z) + D_3(k_x, \omega)(2k_x\beta_3\mu_3)\exp(-i\beta_3 z). \tag{484}$$

Figure 19 is a plot of the displacement in the z-direction (normal) divided by normal pressure versus wavenumber in the x-direction at a forcing frequency of 500 Hz evaluated at $z = d$. In this example, the sum of the thickness of the three plates is very small ($h = 0.045$ m). The solid line represents the thick plate theory using three plates (equation (484)) and the x's represent the thin plate theory (equation (68)) using $h = 0.045$ m. Additionally, the material properties of the three (thick) plates are identical so that a direct comparison between the thick and thin plate theories can be made using only a single thin plate. Parameters used to formulate this model are: bottom of plate 1 (a) is -0.045 m, intersection of plates 2 and 1 (b) is -0.030 m, intersection of plates 3 and 2 (c) is -0.015 m, top of plate 3 (d) is 0 m, Young's modulus (E) is 13.2e9 (1+0.03i) N/m^2, Poisson's ratio (v) is 0.30, and density of the plate (ρ) is 1938 kg/m^3. Only positive values of wavenumber are shown, as the function is symmetric about $k = 0$. Note that there is almost complete agreement between the thick plate theory using three similar thick plates and the thin plate theory using one plate for this specific example. Figure 20 is a plot of the displacement in the z-direction (normal) divided by normal pressure versus wavenumber in the x-direction at a forcing frequency of 500 Hz evaluated at $z = d$. In this example, the sum of the thickness of the three plates is very small ($h = 0.0032$ m). The solid line represents the thick plate theory using three plates (equation (484)) and the x's represent the thin plate theory (equation (68)) using $h = 0.0030$ m. Additionally, the material properties of the three (thick) plates are extremely dissimilar so that a direct comparison between the thick and thin plate theories can be made using only a single thin plate with the thickness of the stiff thick plate ($h = 0.0030$ m). Parameters used to formulate this model are: bottom of plate 1(a) is -0.0032 m, intersection of plates 2 and 1 (b) is -0.0031 m, intersection of plates 3 and 2 (c) is -0.0030 m, top of plate 3 (d) is 0 m, Young's modulus of plate 3 (E_3) is 13.2e9 (1+0.03i) N/m^2, Young's modulus of plates 2 and 1 (E_2 and E_1) is 13.2e4 N/m^2, Poisson's ratio of plates 1, 2, and 3 (v) is 0.30, and density of plates 1, 2, and 3 (ρ) is 1938 kg/m^3. Only positive values of wavenumber are shown, as the function is symmetric about $k = 0$. Note that there is almost complete agreement between the thick plate theory using three dissimilar thick plates and the thin plate theory using one plate (matched to the stiff plate parameters) for this specific example. Figure 21 is a plot of the wavenumber-frequency ($k\omega$) surface of the system using the thick plate equations of motion (equation (484)) evaluated at $z = d$. The first plate is a stiff material and the parameters of this plate are: bottom of the plate (a) is -0.08 m, top of the plate (b) is -0.06 m, Young's modulus of the plate (E_1) is 4.55e10 (1+0.03i) N/m^2, Poisson's ratio of the plate (v_1) is 0.30, and density of the plate (ρ_1) is 7700 kg/m^3. The second plate is a soft material and the parameters of this plate are: bottom of the plate (b) is -0.06 m, top of the plate (c) is -0.01 m, Young's modulus of the plate (E_2) is 1e9(1+0.15i) N/m^2, Poisson's ratio of the plate (v_2) is 0.45, and density of the plate (ρ_2) is 1200 kg/m^3. The third plate is a stiff material and the parameters of this plate are: bottom of the plate (c) is -0.01 m, top of the plate (d) is 0 m, Young's modulus of the plate (E_3) is 13.2e9 (1+0.03i) N/m^2, Poisson's ratio of the plate (v_3) is 0.30, and density of the plate (ρ_3) is 1938 kg/m^3. The color scale to the right of the plot is in decibels. All values greater than -190 dB are displayed as -190 dB and all values less than -220 dB are displayed as -220 dB.

*Figure 19. Transfer Function of a Triple Plate with No Fluid Load
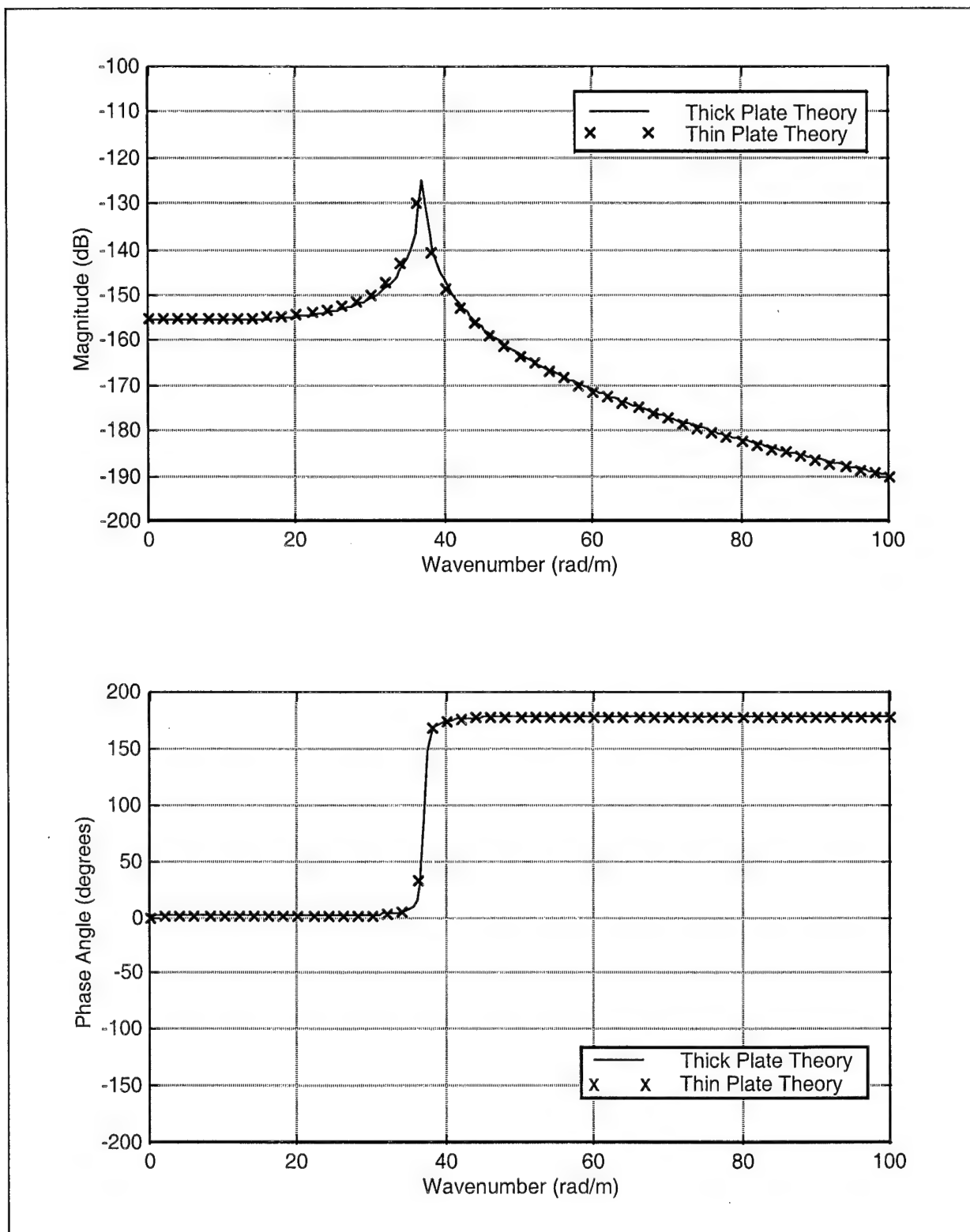(Equation (481)) Using Similar Material Properties at 500 Hz*

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*Figure 20. Transfer Function of a Triple Plate with No Fluid Load
(Equation (481)) Using Dissimilar Material Properties at 500 Hz*
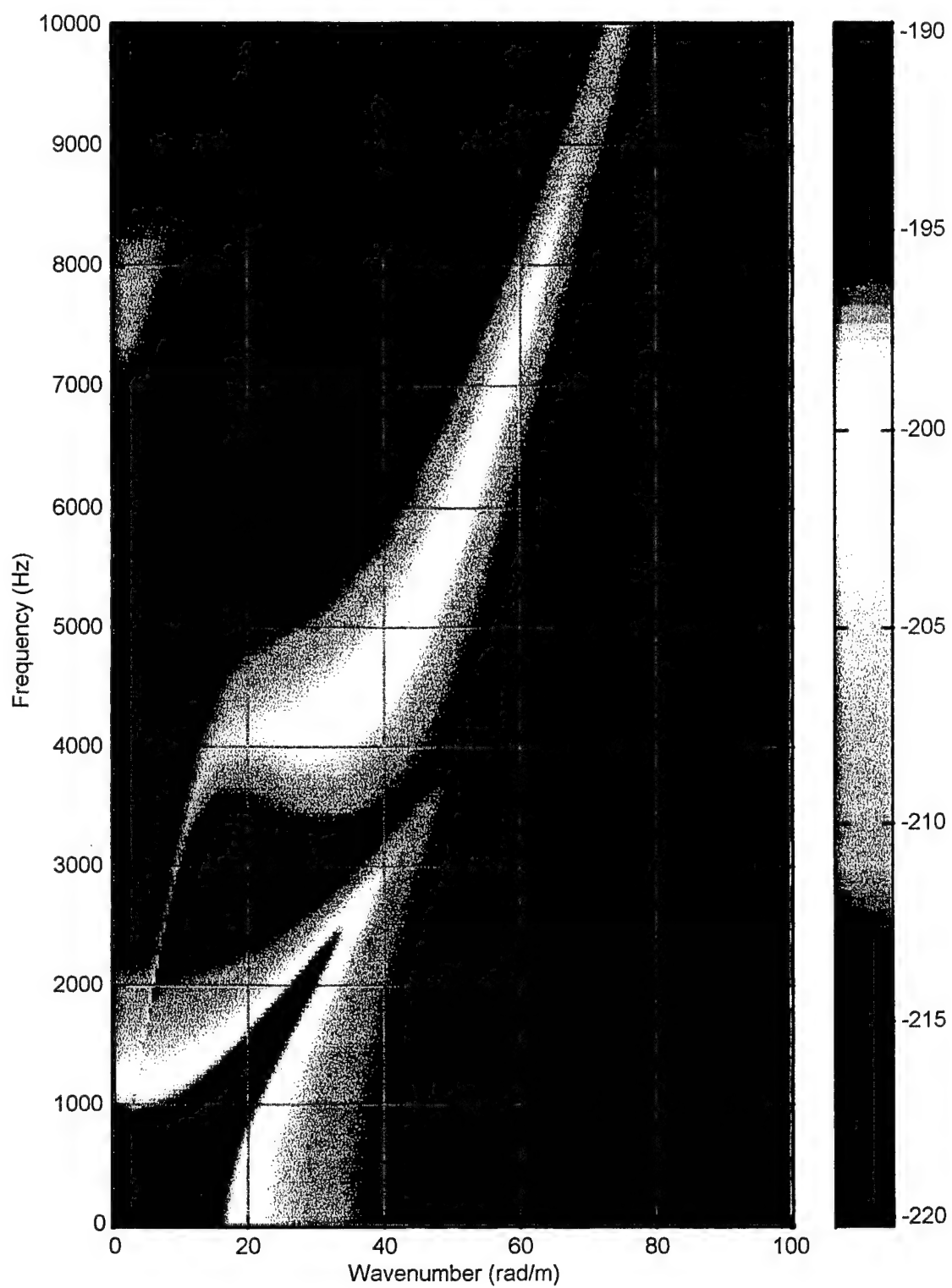
Figure 21. Transfer Surface of a Triple Plate with Fluid No Load

8. TRIPLE PLATE WITH A FLUID LOAD

The sixth transfer function derived is a triple plate with a fluid load. The applied load is modeled as an incident pressure wave in the fluid at definite wavenumber and frequency at location $z = d$, as shown in figure 22. The normal and tangential stresses in the system at the boundary $z = d$ are

$$\tau_{zz}(x,d,t) = (\lambda_3 + 2\mu_3)\frac{\partial u_z(x,d,t)}{\partial z} + \lambda_3 \frac{\partial u_x(x,d,t)}{\partial x} = -p_a(x,d,t), \tag{485}$$

and

$$\tau_{zx}(x,d,t) = \mu_3\left[\frac{\partial u_x(x,d,t)}{\partial z} + \frac{\partial u_z(x,d,t)}{\partial x}\right] = 0, \tag{486}$$

where the subscript 3 denotes plate 3. The interface between plates 3 and 2 requires four equations. The first two are displacement constraints, which are

$$u_z(x,c,t)\big|_{\text{plate3}} = u_z(x,c,t)\big|_{\text{plate2}}, \tag{487}$$

and

$$u_x(x,c,t)\big|_{\text{plate3}} = u_x(x,c,t)\big|_{\text{plate2}}. \tag{488}$$

The second two are stress constraints, which are

$$\tau_{zz}(x,c,t)\big|_{\text{plate3}} = \tau_{zz}(x,c,t)\big|_{\text{plate2}}, \tag{489}$$

and

$$\tau_{zx}(x,c,t)\big|_{\text{plate3}} = \tau_{zx}(x,c,t)\big|_{\text{plate2}}. \tag{490}$$

The interface between plates 2 and 1 requires four equations. The first two are displacement constraints, which are

$$u_z(x,b,t)\big|_{\text{plate2}} = u_z(x,b,t)\big|_{\text{plate1}}, \tag{491}$$

and

$$u_x(x,b,t)\big|_{\text{plate2}} = u_x(x,b,t)\big|_{\text{plate1}}. \tag{492}$$

The second two are stress constraints, which are

$$\tau_{zz}(x,b,t)\big|_{\text{plate2}} = \tau_{zz}(x,b,t)\big|_{\text{plate1}}, \tag{493}$$
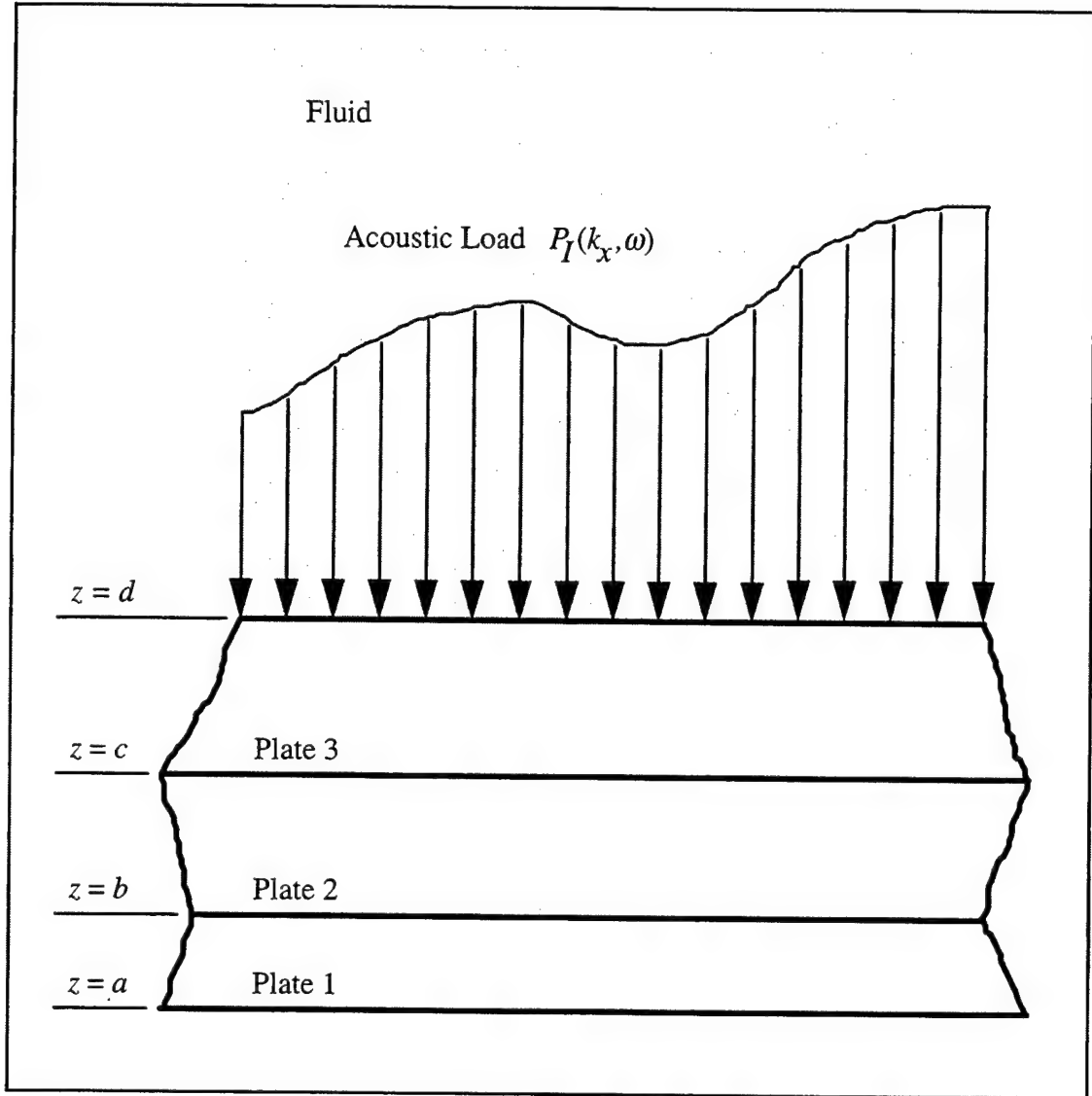
Figure 22. Triple Plate with Fluid Load

and

$$\tau_{zx}(x,b,t)\big|_{\text{plate 2}} = \tau_{zx}(x,b,t)\big|_{\text{plate 1}} \ . \tag{494}$$

Finally, the normal and tangential stresses in the system at the boundary $z = a$ are

$$\tau_{zz}(x,a,t) = (\lambda_1 + 2\mu_1)\frac{\partial u_z(x,a,t)}{\partial z} + \lambda_1 \frac{\partial u_x(x,a,t)}{\partial x} = 0 \ , \tag{495}$$

and

$$\tau_{zx}(x,a,t) = \mu_1\left[\frac{\partial u_x(x,a,t)}{\partial z} + \frac{\partial u_z(x,a,t)}{\partial x}\right] = 0 \ , \tag{496}$$

where the subscript 1 denotes plate 1.

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Combining equations (75) - (78) and (485) - (496) yields the twelve-by-twelve linear system of equations

$$\mathbf{Ax} = \mathbf{b}, \tag{497}$$

where the entries of equation (497) are

$$A_{1,1} = \left(-\alpha_3^2 \lambda_3 - 2\alpha_3^2 \mu_3 - \lambda_3 k_x^2 + \frac{\rho_f \omega^2 \alpha_3}{\gamma} \right) \exp(i\alpha_3 d), \tag{498}$$

$$A_{1,2} = \left(-\alpha_3^2 \lambda_3 - 2\alpha_3^2 \mu_3 - \lambda_3 k_x^2 - \frac{\rho_f \omega^2 \alpha_3}{\gamma} \right) \exp(-i\alpha_3 d), \tag{499}$$

$$A_{1,3} = \left(-2k_x \beta_3 \mu_3 + \frac{\rho_f \omega^2 k_x}{\gamma} \right) \exp(i\beta_3 d), \tag{500}$$

$$A_{1,4} = \left(2k_x \beta_3 \mu_3 + \frac{\rho_f \omega^2 k_x}{\gamma} \right) \exp(-i\beta_3 d), \tag{501}$$

$$A_{1,5} = 0, \tag{502}$$

$$A_{1,6} = 0, \tag{503}$$

$$A_{1,7} = 0, \tag{504}$$

$$A_{1,8} = 0, \tag{505}$$

$$A_{1,9} = 0, \tag{506}$$

$$A_{1,10} = 0, \tag{507}$$

$$A_{1,11} = 0, \tag{508}$$

$$A_{1,12} = 0, \tag{509}$$

$$A_{2,1} = \left(-2\mu_3 k_x \alpha_3 \right) \exp(i\alpha_3 d), \tag{510}$$

$$A_{2,2} = \left(2\mu_3 k_x \alpha_3 \right) \exp(-i\alpha_3 d), \tag{511}$$

$$A_{2,3} = \left(\mu_3 \beta_3^2 - \mu_3 k_x^2\right)\exp(i\beta_3 d), \tag{512}$$

$$A_{2,4} = \left(\mu_3 \beta_3^2 - \mu_3 k_x^2\right)\exp(-i\beta_3 d), \tag{513}$$

$$A_{2,5} = 0, \tag{514}$$

$$A_{2,6} = 0, \tag{515}$$

$$A_{2,7} = 0, \tag{516}$$

$$A_{2,8} = 0, \tag{517}$$

$$A_{2,9} = 0, \tag{518}$$

$$A_{2,10} = 0, \tag{519}$$

$$A_{2,11} = 0, \tag{520}$$

$$A_{2,12} = 0, \tag{521}$$

$$A_{3,1} = \left(i\alpha_3\right)\exp(i\alpha_3 c), \tag{522}$$

$$A_{3,2} = \left(-i\alpha_3\right)\exp(-i\alpha_3 c), \tag{523}$$

$$A_{3,3} = \left(ik_x\right)\exp(i\beta_3 c), \tag{524}$$

$$A_{3,4} = \left(ik_x\right)\exp(-i\beta_3 c), \tag{525}$$

$$A_{3,5} = \left(-i\alpha_2\right)\exp(i\alpha_2 c), \tag{526}$$

$$A_{3,6} = \left(i\alpha_2\right)\exp(-i\alpha_2 c), \tag{527}$$

$$A_{3,7} = \left(-ik_x\right)\exp(i\beta_2 c), \tag{528}$$

$$A_{3,8} = \left(-ik_x\right)\exp(-i\beta_2 c), \tag{529}$$

$$A_{3,9} = 0, \tag{530}$$

$$A_{3,10} = 0 , \tag{531}$$

$$A_{3,11} = 0 , \tag{532}$$

$$A_{3,12} = 0 , \tag{533}$$

$$A_{4,1} = (ik_x)\exp(i\alpha_3 c) , \tag{534}$$

$$A_{4,2} = (ik_x)\exp(-i\alpha_3 c) , \tag{535}$$

$$A_{4,3} = (-i\beta_3)\exp(i\beta_3 c) , \tag{536}$$

$$A_{4,4} = (i\beta_3)\exp(-i\beta_3 c) , \tag{537}$$

$$A_{4,5} = (-ik_x)\exp(i\alpha_2 c) , \tag{538}$$

$$A_{4,6} = (-ik_x)\exp(-i\alpha_2 c) , \tag{539}$$

$$A_{4,7} = (i\beta_2)\exp(i\beta_2 c) , \tag{540}$$

$$A_{4,8} = (-i\beta_2)\exp(-i\beta_2 c) , \tag{541}$$

$$A_{4,9} = 0 , \tag{542}$$

$$A_{4,10} = 0 , \tag{543}$$

$$A_{4,11} = 0 , \tag{544}$$

$$A_{4,12} = 0 , \tag{545}$$

$$A_{5,1} = \left(-\alpha_3^2\lambda_3 - 2\alpha_3^2\mu_3 - \lambda_3 k_x^2\right)\exp(i\alpha_3 c) , \tag{546}$$

$$A_{5,2} = \left(-\alpha_3^2\lambda_3 - 2\alpha_3^2\mu_3 - \lambda_3 k_x^2\right)\exp(-i\alpha_3 c) , \tag{547}$$

$$A_{5,3} = \left(-2k_x\beta_3\mu_3\right)\exp(i\beta_3 c) , \tag{548}$$

$$A_{5,4} = \left(2k_x\beta_3\mu_3\right)\exp(-i\beta_3 c) , \tag{549}$$

$$A_{5,5} = \left(\alpha_2^2 \lambda_2 + 2\alpha_2^2 \mu_2 + \lambda_2 k_x^2\right)\exp(\mathrm{i}\alpha_2 c)\,, \tag{550}$$

$$A_{5,6} = \left(\alpha_2^2 \lambda_2 + 2\alpha_2^2 \mu_2 + \lambda_2 k_x^2\right)\exp(-\mathrm{i}\alpha_2 c)\,, \tag{551}$$

$$A_{5,7} = \left(2k_x \beta_2 \mu_2\right)\exp(\mathrm{i}\beta_2 c)\,, \tag{552}$$

$$A_{5,8} = \left(-2k_x \beta_2 \mu_2\right)\exp(-\mathrm{i}\beta_2 c)\,, \tag{553}$$

$$A_{5,9} = 0\,, \tag{554}$$

$$A_{5,10} = 0\,, \tag{555}$$

$$A_{5,11} = 0\,, \tag{556}$$

$$A_{5,12} = 0\,, \tag{557}$$

$$A_{6,1} = \left(-2\mu_3 k_x \alpha_3\right)\exp(\mathrm{i}\alpha_3 c)\,, \tag{558}$$

$$A_{6,2} = \left(2\mu_3 k_x \alpha_3\right)\exp(-\mathrm{i}\alpha_3 c)\,, \tag{559}$$

$$A_{6,3} = \left(\mu_3 \beta_3^2 - \mu_3 k_x^2\right)\exp(\mathrm{i}\beta_3 c)\,, \tag{560}$$

$$A_{6,4} = \left(\mu_3 \beta_3^2 - \mu_3 k_x^2\right)\exp(-\mathrm{i}\beta_3 c)\,, \tag{561}$$

$$A_{6,5} = \left(2\mu_2 k_x \alpha_2\right)\exp(\mathrm{i}\alpha_2 c)\,, \tag{562}$$

$$A_{6,6} = \left(-2\mu_2 k_x \alpha_2\right)\exp(-\mathrm{i}\alpha_2 c)\,, \tag{563}$$

$$A_{6,7} = \left(-\mu_2 \beta_2^2 + \mu_2 k_x^2\right)\exp(\mathrm{i}\beta_2 c)\,, \tag{564}$$

$$A_{6,8} = \left(-\mu_2 \beta_2^2 + \mu_2 k_x^2\right)\exp(-\mathrm{i}\beta_2 c)\,, \tag{565}$$

$$A_{6,9} = 0\,, \tag{566}$$

$$A_{6,10} = 0\,, \tag{567}$$

$$A_{6,11} = 0\,, \tag{568}$$

$$A_{6,12} = 0,$$

(569)

$$A_{7,1} = 0,$$

(570)

$$A_{7,2} = 0,$$

(571)

$$A_{7,3} = 0,$$

(572)

$$A_{7,4} = 0,$$

(573)

$$A_{7,5} = (i\alpha_2)\exp(i\alpha_2 b),$$

(574)

$$A_{7,6} = (-i\alpha_2)\exp(-i\alpha_2 b),$$

(575)

$$A_{7,7} = (ik_x)\exp(i\beta_2 b),$$

(576)

$$A_{7,8} = (ik_x)\exp(-i\beta_2 b),$$

(577)

$$A_{7,9} = (-i\alpha_1)\exp(i\alpha_1 b),$$

(578)

$$A_{7,10} = (i\alpha_1)\exp(-i\alpha_1 b),$$

(579)

$$A_{7,11} = (-ik_x)\exp(i\beta_1 b),$$

(580)

$$A_{7,12} = (-ik_x)\exp(-i\beta_1 b),$$

(581)

$$A_{8,1} = 0,$$

(582)

$$A_{8,2} = 0,$$

(583)

$$A_{8,3} = 0,$$

(584)

$$A_{8,4} = 0,$$

(585)

$$A_{8,5} = (ik_x)\exp(i\alpha_2 b),$$

(586)

$$A_{8,6} = (ik_x)\exp(-i\alpha_2 b),$$

(587)

$$A_{8,7} = (-i\beta_2)\exp(i\beta_2 b),$$

(588)

$$A_{8,8} = (i\beta_2)\exp(-i\beta_2 b), \tag{589}$$

$$A_{8,9} = (-ik_x)\exp(i\alpha_1 b), \tag{590}$$

$$A_{8,10} = (-ik_x)\exp(-i\alpha_1 b), \tag{591}$$

$$A_{8,11} = (i\beta_1)\exp(i\beta_1 b), \tag{592}$$

$$A_{8,12} = (-i\beta_1)\exp(-i\beta_1 b), \tag{593}$$

$$A_{9,1} = 0, \tag{594}$$

$$A_{9,2} = 0, \tag{595}$$

$$A_{9,3} = 0, \tag{596}$$

$$A_{9,4} = 0, \tag{597}$$

$$A_{9,5} = \left(-\alpha_2^2\lambda_2 - 2\alpha_2^2\mu_2 - \lambda_2 k_x^2\right)\exp(i\alpha_2 b), \tag{598}$$

$$A_{9,6} = \left(-\alpha_2^2\lambda_2 - 2\alpha_2^2\mu_2 - \lambda_2 k_x^2\right)\exp(-i\alpha_2 b), \tag{599}$$

$$A_{9,7} = \left(-2k_x\beta_2\mu_2\right)\exp(i\beta_2 b), \tag{600}$$

$$A_{9,8} = \left(2k_x\beta_2\mu_2\right)\exp(-i\beta_2 b), \tag{601}$$

$$A_{9,9} = \left(\alpha_1^2\lambda_1 + 2\alpha_1^2\mu_1 + \lambda_1 k_x^2\right)\exp(i\alpha_1 b), \tag{602}$$

$$A_{9,10} = \left(\alpha_1^2\lambda_1 + 2\alpha_1^2\mu_1 + \lambda_1 k_x^2\right)\exp(-i\alpha_1 b), \tag{603}$$

$$A_{9,11} = \left(2k_x\beta_1\mu_1\right)\exp(i\beta_1 b), \tag{604}$$

$$A_{9,12} = \left(-2k_x\beta_1\mu_1\right)\exp(-i\beta_1 b), \tag{605}$$

$$A_{10,1} = 0, \tag{606}$$

$$A_{10,2} = 0, \tag{607}$$

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$$A_{10,3} = 0, \tag{608}$$

$$A_{10,4} = 0, \tag{609}$$

$$A_{10,5} = \left(-2\mu_2 k_x \alpha_2\right)\exp(i\alpha_2 b), \tag{610}$$

$$A_{10,6} = \left(2\mu_2 k_x \alpha_2\right)\exp(-i\alpha_2 b), \tag{611}$$

$$A_{10,7} = \left(\mu_2 \beta_2^2 - \mu_2 k_x^2\right)\exp(i\beta_2 b), \tag{612}$$

$$A_{10,8} = \left(\mu_2 \beta_2^2 - \mu_2 k_x^2\right)\exp(-i\beta_2 b), \tag{613}$$

$$A_{10,9} = \left(2\mu_1 k_x \alpha_1\right)\exp(i\alpha_1 b), \tag{614}$$

$$A_{10,10} = \left(-2\mu_1 k_x \alpha_1\right)\exp(-i\alpha_1 b), \tag{615}$$

$$A_{10,11} = \left(-\mu_1 \beta_1^2 + \mu_1 k_x^2\right)\exp(i\beta_1 b), \tag{616}$$

$$A_{10,12} = \left(-\mu_1 \beta_1^2 + \mu_1 k_x^2\right)\exp(-i\beta_1 b), \tag{617}$$

$$A_{11,1} = 0, \tag{618}$$

$$A_{11,2} = 0, \tag{619}$$

$$A_{11,3} = 0, \tag{620}$$

$$A_{11,4} = 0, \tag{621}$$

$$A_{11,5} = 0, \tag{622}$$

$$A_{11,6} = 0, \tag{623}$$

$$A_{11,7} = 0, \tag{624}$$

$$A_{11,8} = 0, \tag{625}$$

$$A_{11,9} = \left(-\alpha_1^2 \lambda_1 - 2\alpha_1^2 \mu_1 - \lambda_1 k_x^2\right)\exp(i\alpha_1 a), \tag{626}$$

$$A_{11,10} = \left(-\alpha_1^2\lambda_1 - 2\alpha_1^2\mu_1 - \lambda_1 k_x^2\right)\exp(-i\alpha_1 a), \tag{627}$$

$$A_{11,11} = \left(-2k_x\beta_1\mu_1\right)\exp(i\beta_1 a), \tag{628}$$

$$A_{11,12} = \left(2k_x\beta_1\mu_1\right)\exp(-i\beta_1 a), \tag{629}$$

$$A_{12,1} = 0, \tag{630}$$

$$A_{12,2} = 0, \tag{631}$$

$$A_{12,3} = 0, \tag{632}$$

$$A_{12,4} = 0, \tag{633}$$

$$A_{12,5} = 0, \tag{634}$$

$$A_{12,6} = 0, \tag{635}$$

$$A_{12,7} = 0, \tag{636}$$

$$A_{12,8} = 0, \tag{637}$$

$$A_{12,9} = \left(-2\mu_1 k_x\alpha_1\right)\exp(i\alpha_1 a), \tag{638}$$

$$A_{12,10} = \left(2\mu_1 k_x\alpha_1\right)\exp(-i\alpha_1 a), \tag{639}$$

$$A_{12,11} = \left(\mu_1\beta_1^2 - \mu_1 k_x^2\right)\exp(i\beta_1 a), \tag{640}$$

$$A_{12,12} = \left(\mu_1\beta_1^2 - \mu_1 k_x^2\right)\exp(-i\beta_1 a), \tag{641}$$

$$x_{1,1} = A_3(k_x,\omega), \tag{642}$$

$$x_{2,1} = B_3(k_x,\omega), \tag{643}$$

$$x_{3,1} = C_3(k_x,\omega), \tag{644}$$

$$x_{4,1} = D_3(k_x,\omega), \tag{645}$$

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$$x_{5,1} = A_2(k_x, \omega) \,, \tag{646}$$

$$x_{6,1} = B_2(k_x, \omega) \,, \tag{647}$$

$$x_{7,1} = C_2(k_x, \omega) \,, \tag{648}$$

$$x_{8,1} = D_2(k_x, \omega) \,, \tag{649}$$

$$x_{9,1} = A_1(k_x, \omega) \,, \tag{650}$$

$$x_{10,1} = B_1(k_x, \omega) \,, \tag{651}$$

$$x_{11,1} = C_1(k_x, \omega) \,, \tag{652}$$

$$x_{12,1} = D_1(k_x, \omega) \,, \tag{653}$$

$$b_{1,1} = -2P_I(k_x, \omega) \exp(-i\gamma d) \,, \tag{654}$$

$$b_{2,1} = 0 \,, \tag{655}$$

$$b_{3,1} = 0 \,, \tag{656}$$

$$b_{4,1} = 0 \,, \tag{657}$$

$$b_{5,1} = 0 \,, \tag{658}$$

$$b_{6,1} = 0 \,, \tag{659}$$

$$b_{7,1} = 0 \,, \tag{660}$$

$$b_{8,1} = 0 \,, \tag{661}$$

$$b_{9,1} = 0 \,, \tag{662}$$

$$b_{10,1} = 0 \,, \tag{663}$$

$$b_{11,1} = 0 \,, \tag{664}$$

and

$$b_{12,1} = 0. \tag{665}$$

Using equations (498) - (665), the solution to the constants A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2, A_3, B_3, C_3, and D_3 can be found by

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}. \tag{666}$$

Additionally, the transfer function between the normal displacement at location z (when $z < b$) and the structural load can be written as

$$T(k_x, \omega) = \frac{U_z(z, k_x, \omega)}{P_I(k_x, \omega)\exp(-i\gamma d)} = A_1(k_x, \omega)i\alpha_1 \exp(i\alpha_1 z) -$$

$$B_1(k_x, \omega)i\alpha_1 \exp(-i\alpha_1 z) +$$

$$C_1(k_x, \omega)ik_x \exp(i\beta_1 z) + D_1(k_x, \omega)ik_x \exp(-i\beta_1 z), \tag{667}$$

or (when $z > b$ and $z < c$)

$$T(k_x, \omega) = \frac{U_z(z, k_x, \omega)}{P_I(k_x, \omega)\exp(-i\gamma d)} = A_2(k_x, \omega)i\alpha_2 \exp(i\alpha_2 z) -$$

$$B_2(k_x, \omega)i\alpha_2 \exp(-i\alpha_2 z) +$$

$$C_2(k_x, \omega)ik_x \exp(i\beta_2 z) + D_2(k_x, \omega)ik_x \exp(-i\beta_2 z), \tag{668}$$

of (when $z > c$)

$$T(k_x, \omega) = \frac{U_z(z, k_x, \omega)}{P_I(k_x, \omega)\exp(-i\gamma d)} = A_3(k_x, \omega)i\alpha_3 \exp(i\alpha_3 z) -$$

$$B_3(k_x, \omega)i\alpha_3 \exp(-i\alpha_3 z) +$$

$$C_3(k_x, \omega)ik_x \exp(i\beta_3 z) + D_3(k_x, \omega)ik_x \exp(-i\beta_3 z). \tag{669}$$

The transfer function between the normal stress at location z (when $z < b$) and the structural load can be written as

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$$T(k_x, \omega) = \frac{T_{zz}(z, k_x, \omega)}{P_I(k_x, \omega)\exp(-i\gamma d)} = A_1(k_x, \omega)\left(-\alpha_1^2\lambda_1 - 2\alpha_1^2\mu_1 - \lambda_1 k_x^2\right)\exp(i\alpha_1 z) +$$

$$B_1(k_x, \omega)\left(-\alpha_1^2\lambda_1 - 2\alpha_1^2\mu_1 - \lambda_1 k_x^2\right)\exp(-i\alpha_1 z) +$$

$$C_1(k_x, \omega)(-2k_x\beta_1\mu_1)\exp(i\beta_1 z) + D_1(k_x, \omega)(2k_x\beta_1\mu_1)\exp(-i\beta_1 z), \qquad (670)$$

or (when $z > b$ and $z < c$)

$$T(k_x, \omega) = \frac{T_{zz}(z, k_x, \omega)}{P_I(k_x, \omega)\exp(-i\gamma d)} = A_2(k_x, \omega)\left(-\alpha_2^2\lambda_2 - 2\alpha_2^2\mu_2 - \lambda_2 k_x^2\right)\exp(i\alpha_2 z) +$$

$$B_2(k_x, \omega)\left(-\alpha_2^2\lambda_2 - 2\alpha_2^2\mu_2 - \lambda_2 k_x^2\right)\exp(-i\alpha_2 z) +$$

$$C_2(k_x, \omega)(-2k_x\beta_2\mu_2)\exp(i\beta_2 z) + D_2(k_x, \omega)(2k_x\beta_2\mu_2)\exp(-i\beta_2 z), \qquad (671)$$

of (when $z > c$)

$$T(k_x, \omega) = \frac{T_{zz}(z, k_x, \omega)}{P_I(k_x, \omega)\exp(-i\gamma d)} = A_3(k_x, \omega)\left(-\alpha_3^2\lambda_3 - 2\alpha_3^2\mu_3 - \lambda_3 k_x^2\right)\exp(i\alpha_3 z) +$$

$$B_3(k_x, \omega)\left(-\alpha_3^2\lambda_3 - 2\alpha_3^2\mu_3 - \lambda_3 k_x^2\right)\exp(-i\alpha_3 z) +$$

$$C_3(k_x, \omega)(-2k_x\beta_3\mu_3)\exp(i\beta_3 z) + D_3(k_x, \omega)(2k_x\beta_3\mu_3)\exp(-i\beta_3 z). \qquad (672)$$

Figure 23 is a plot of displacement in the z-direction (normal) divided by normal pressure versus wavenumber in the x-direction at a forcing frequency of 500 Hz evaluated at $z = d$. In this example, the sum of the thickness of the three plates is very small ($h = 0.045$ m). The solid line represents the thick plate theory using three plates (equation (669)) and the x's represent the thin plate theory (equation (107)) using $h = 0.045$ m. Additionally, the material properties of the three (thick) plates are identical so that a direct comparison between the thick and thin plate theories can be made using only a single thin plate. Parameters used to formulate this model are: bottom of plate 1 (a) is -0.045 m, intersection of plates 2 and 1 (b) is -0.030 m, intersection of plates 3 and 2 (c) is -0.015 m, top of plate 3 (d) is 0 m, Young's modulus (E) is 13.2e9 (1+0.03i) N/m^2, Poisson's ratio (v) is 0.30, and density of the plate (ρ) is 1938 kg/m^3. Only positive values of wavenumber are shown, as the function is symmetric about $k = 0$. Note that there is almost complete agreement between the thick plate theory using three similar thick plates and the thin plate theory using one plate for this specific example. Figure 24 is a plot of the displacement in the z-direction (normal) divided by normal pressure versus wavenumber in the x direction at a forcing frequency of 500 Hz evaluated at $z = d$. In this example, the sum of the thickness of the three plates is very small ($h = 0.0032$ m). The solid line represents the thick plate theory using three plates (equation (669)) and the x's represent the thin plate theory (equation (107)) using $h = 0.0030$ m. Additionally, the material properties of the three (thick) plates are extremely dissimilar so that a direct comparison between the thick and thin plate

theories can be made using only a single thin plate with the thickness of the stiff thick plate ($h =$ 0.0030 m). Parameters used to formulate this model are: bottom of plate 1 (a) is -0.0032 m, intersection of plates 2 and 1 (b) is -0.0031 m, intersection of plates 3 and 2 (c) is -0.0030 m, top of plate 3 (d) is 0 m, Young's modulus of plate 3 (E_3) is 13.2e9 (1+0.03i) N/m^2, Young's modulus of plates 2 and 1 (E_2 and E_1) is 13.2e4 N/m^2, Poisson's ratio of plates 1, 2, and 3 (v) is 0.30, and density of plates 1, 2, and 3 (ρ) is 1938 kg/m^3. Only positive values of wavenumber are shown, as the function is symmetric about $k = 0$. Note that there is almost complete agreement at between the thick plate theory using three dissimilar thick plates and the thin plate theory using one plate (matched to the stiff plate parameters) for this specific example. Figure 25 is a plot of the wavenumber-frequency (kω) surface of the system using the thick plate equations of motion (equation (669)) evaluated at $z = d$. The first plate is a stiff material and the parameters of this plate are: bottom of the plate (a) is -0.08 m, top of the plate (b) is -0.06 m, Young's modulus of the plate (E_1) is 4.55e10 (1+0.03i) N/m^2, Poisson's ratio of the plate (v_1) is 0.30, and density of the plate (ρ_1) is 7700 kg/m^3. The second plate is a soft material and the parameters of this plate are: bottom of the plate (b) is -0.06 m, top of the plate (c) is -0.01 m, Young's modulus of the plate (E_2) is 1e9(1+0.15i) N/m^2, Poisson's ratio of the plate (v_2) is 0.45, and density of the plate (ρ_2) is 1200 kg/m^3. The third plate is a stiff material and the parameters of this plate are: bottom of the plate (c) is -0.01 m, top of the plate (d) is 0 m, Young's modulus of the plate (E_3) is 13.2e9 (1+0.03i) N/m^2, Poisson's ratio of the plate (v_3) is 0.30, and density of the plate (ρ_3) is 1938 kg/m^3. The color scale to the right of the plot is in decibels. All values greater than -190 dB are displayed as -190 dB and all values less than -220 dB are displayed as -220 dB.

Figures 26, 27, and 28 are plots of stress in the z-direction (normal) divided by normal pressure versus wavenumber in the x-direction at forcing frequencies of 1000, 3000, and 6000 Hz evaluated at $z = (b+c)/2$. This evaluation point is the middle of the second plate. The first plate is a stiff material and the parameters of this plate are: bottom of the plate (a) is -0.279 m, top of the plate (b) is -0.203 m, Young's modulus of the plate (E_1) is 2.07e11 N/m^2, Poisson's ratio of the plate (v_1) is 0.30, and density of the plate (ρ_1) is 7830 kg/m^3. The second plate is a soft material and the parameters of this plate are: bottom of the plate (b) is -0.203 m, top of the plate (c) is -0.076 m, Young's modulus of the plate (E_2) is 1e7(1+0.30i) N/m^2, Poisson's ratio of the plate (v_2) is 0.40, and density of the plate (ρ_2) is 1000 kg/m^3. The third plate is a soft material and the parameters of this plate are: bottom of the plate (c) is -0.076 m, top of the plate (d) is 0 m, Young's modulus of the plate (E_3) is 5e7 (1+0.15i) N/m^2, Poisson's ratio of the plate (v_3) is 0.45, and density of the plate (ρ_3) is 1200 kg/m^3. The compressional wavespeed of the fluid (c_f) is 1500 m/s and the density of the fluid (ρ_f) is 1025 kg/m^3.
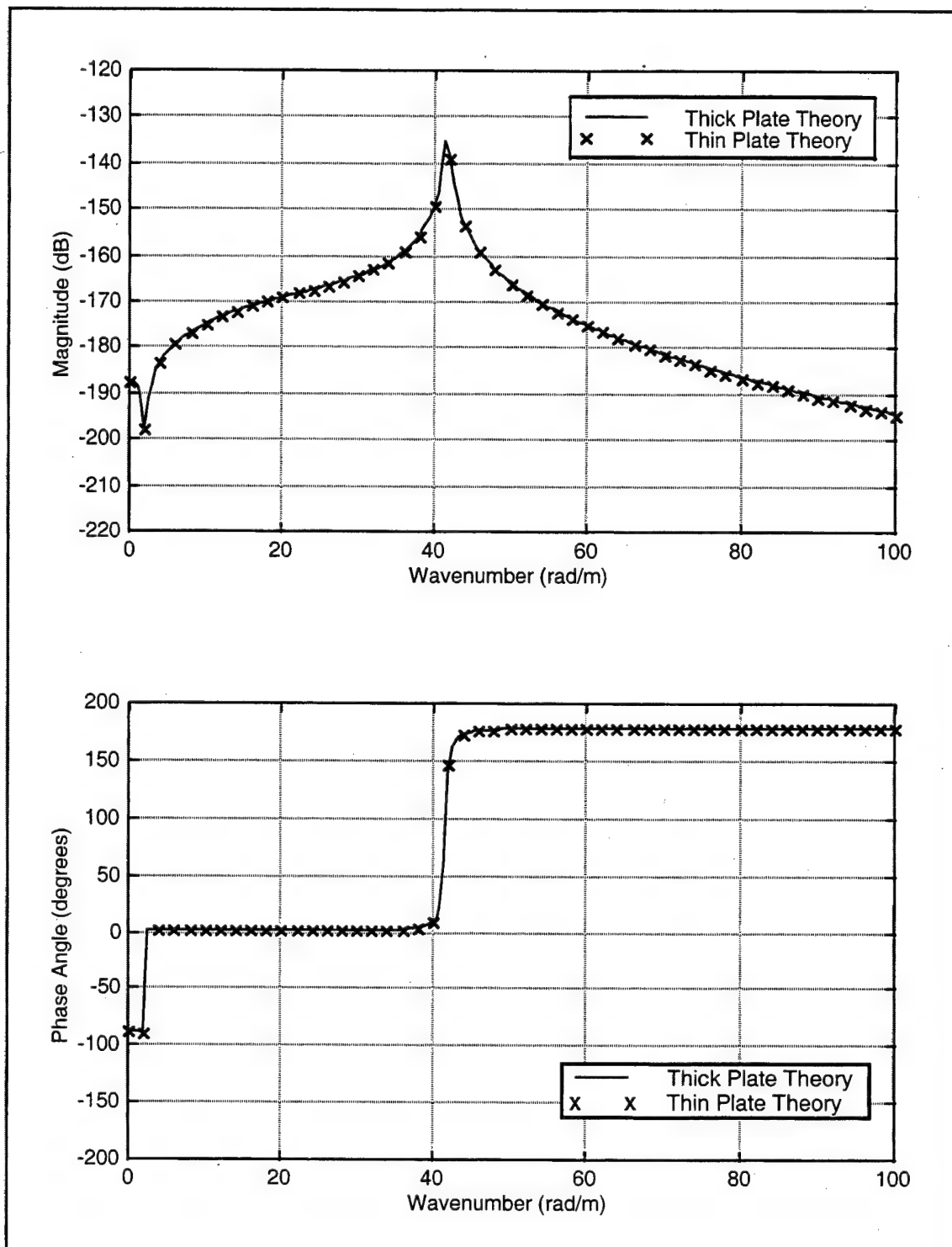
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Figure 23. Transfer Function of a Triple Plate With Fluid Load (Equation (669)) Using Similar Material Properties at 500 Hz
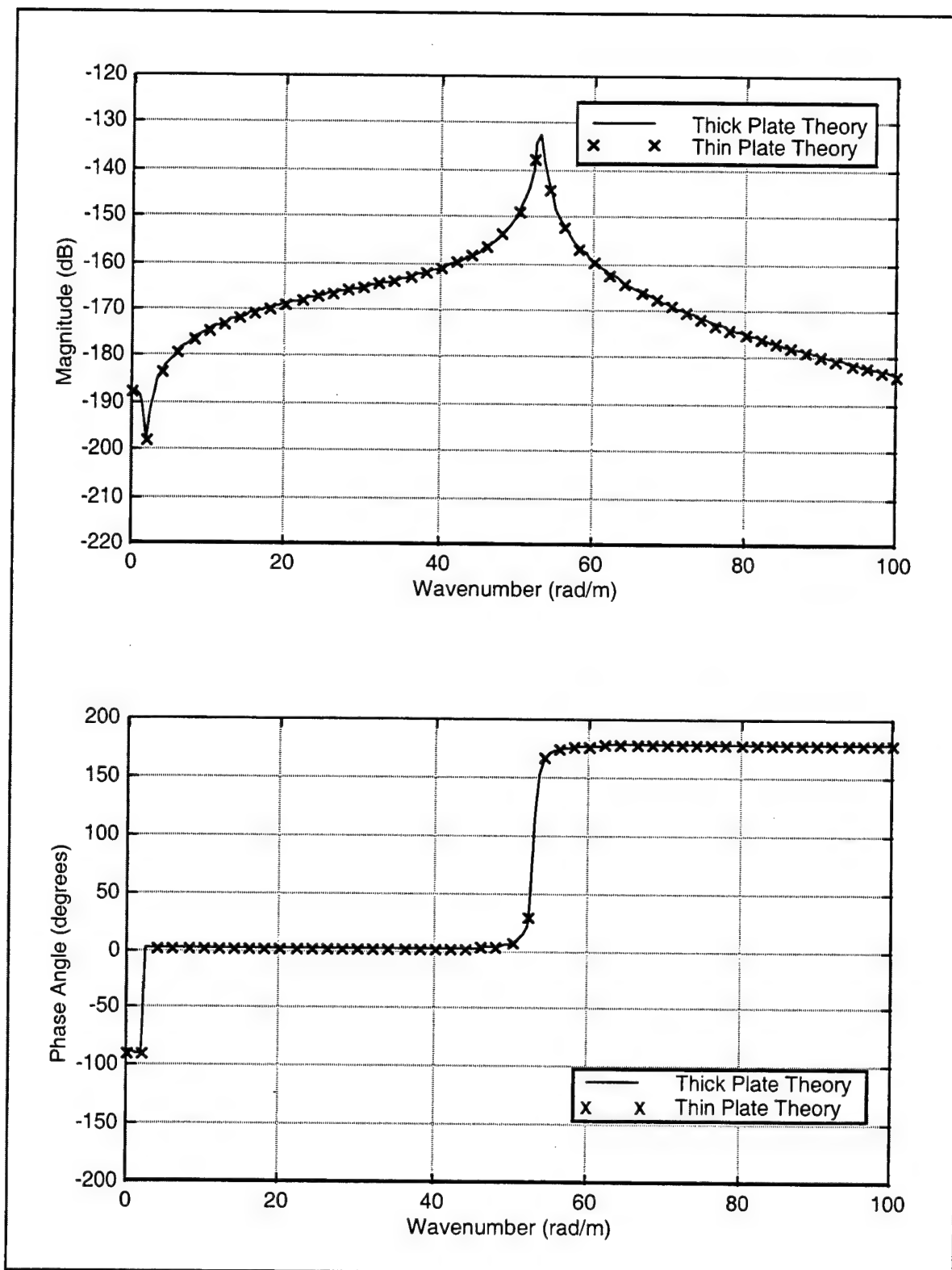
Figure 24. Transfer Function of a Triple Plate with Fluid Load (Equation (669)) Using Dissimilar Material Properties at 500 Hz
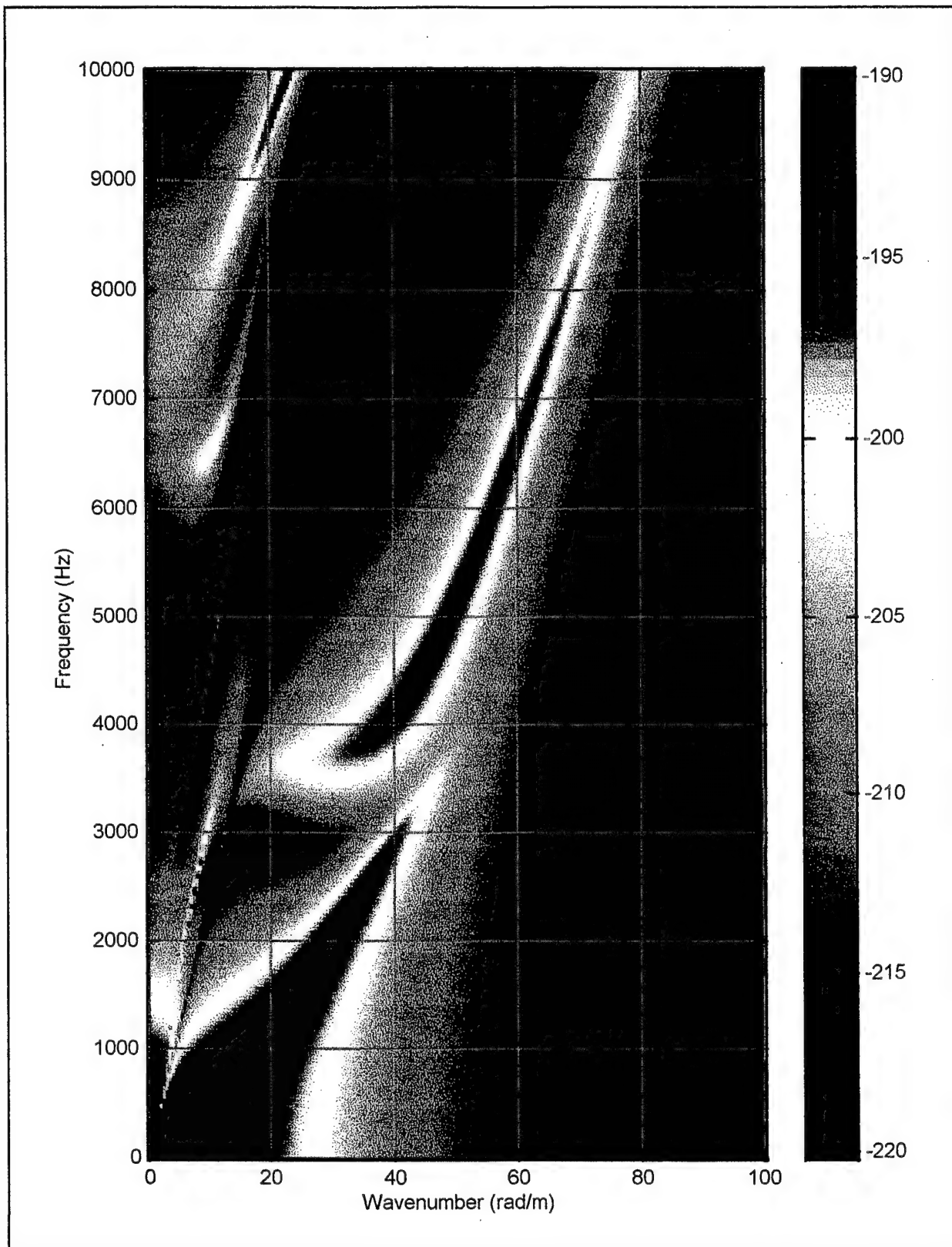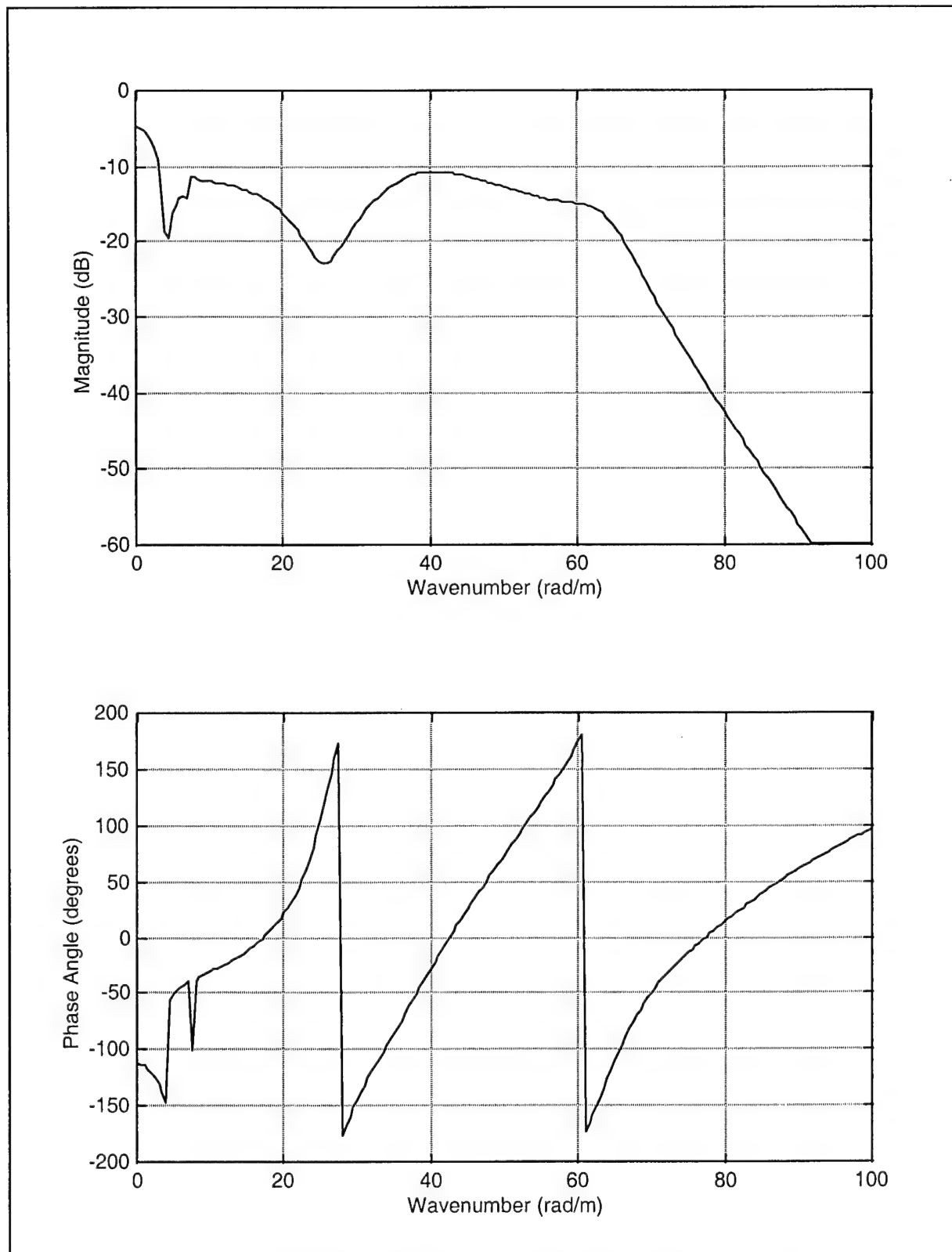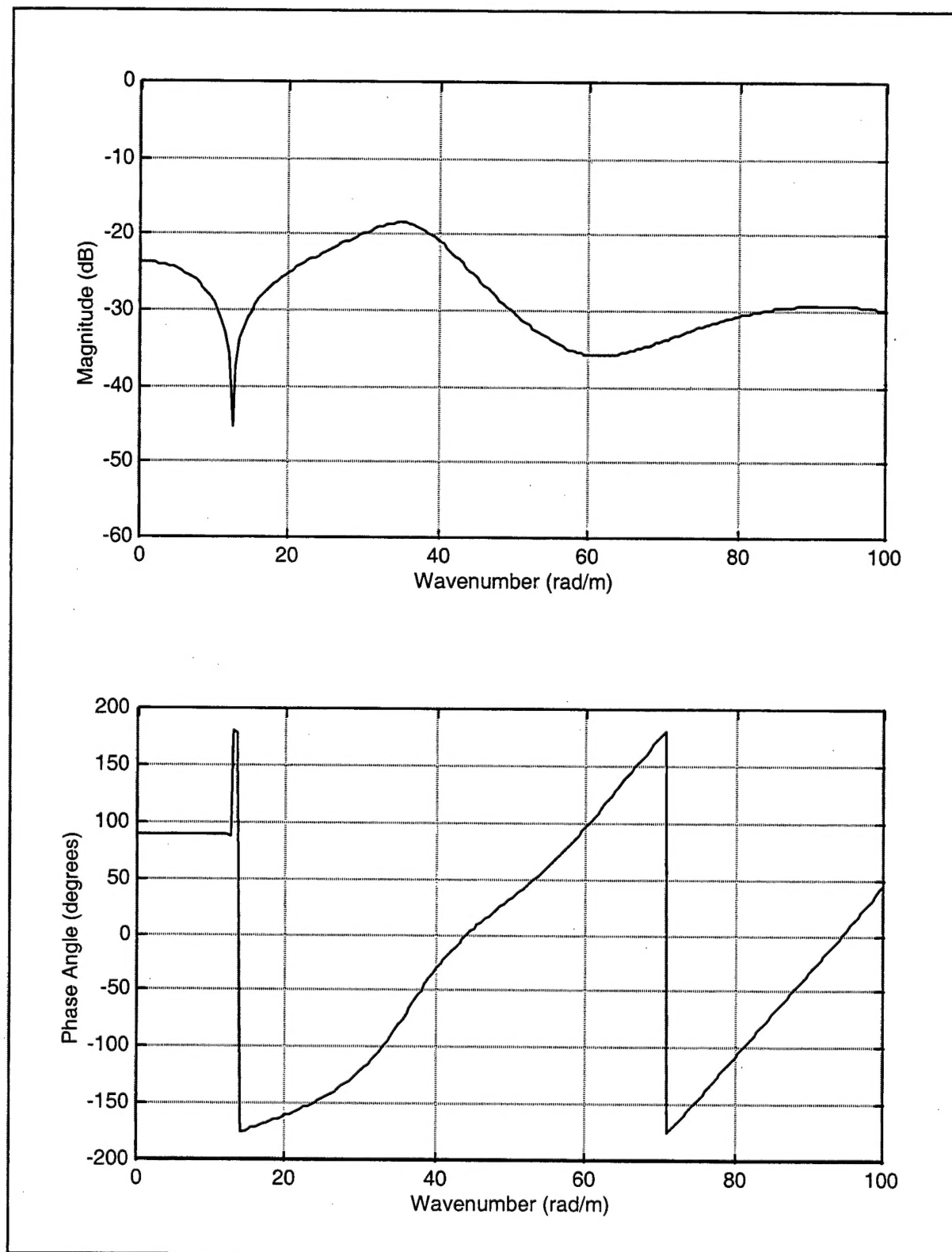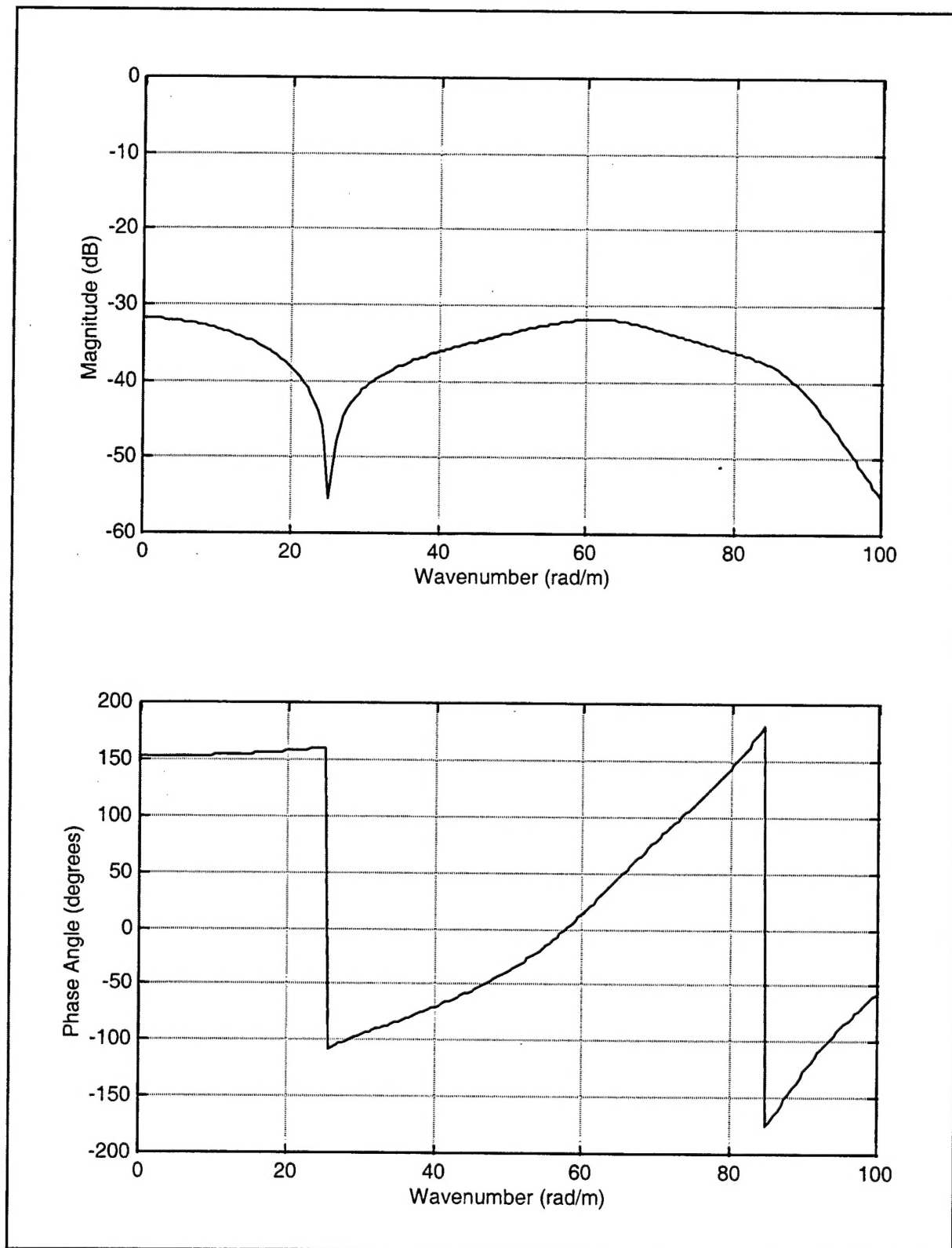
Figure 25. Transfer Surface of a Triple Plate with Fluid Load

Figure 26. Transfer Function of a Triple Plate with Fluid Load
(Equation (671)) at 1000 Hz

Figure 27. Transfer Function of a Triple Plate with Fluid Load (Equation (671)) at 3000 Hz

Figure 28. Transfer Function of a Triple Plate with Fluid Load (Equation (671)) at 6000 Hz

9. SUMMARY

The transfer functions for six different thick plate configurations were derived. This included a single thick plate, a single thick plate with a fluid load, a double thick plate, a double thick plate with a fluid load, a triple thick plate, and a triple thick plate with a fluid load. Validation comparisons were made using similar transfer functions derived with the thin plate theory, the results of which are that for small plate thicknesses, these transfer functions are nearly identical.

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